

DEFORMATION OF C*-ALGEBRAS BY COCYCLES ON LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. Given a C*-algebra A with a left action of a locally compact quantum group G on it and a unitary 2-cocycle Ω on \hat{G} , we define a deformation A_Ω of A . The construction behaves well under certain additional technical assumptions on Ω , the most important of which is regularity, meaning that $C_0(G)_\Omega \rtimes G$ is isomorphic to the algebra of compact operators on some Hilbert space. In particular, then A_Ω is stably isomorphic to the iterated twisted crossed product $\hat{G}^{\text{op}} \rtimes_\Omega G \rtimes A$. Also, in good situations, the C*-algebra A_Ω carries a left action of the deformed quantum group G_Ω and we have an isomorphism $G_\Omega \rtimes A_\Omega \cong G \rtimes A$. When G is a genuine locally compact group, we show that the action of G on $C_0(G)_\Omega = C_r^*(\hat{G}; \Omega)$ is always integrable. Stronger assumptions of properness and saturation of the action imply regularity. As an example, we make a preliminary analysis of the cocycles on the duals of some solvable Lie groups recently constructed by Bieliavsky et al., and discuss the relation of our construction to that of Bieliavsky and Gayral.

INTRODUCTION

Assume $(\mathcal{H}, \hat{\Delta})$ is a Hopf algebra and \mathcal{A} is a left \mathcal{H} -module algebra, with the action of \mathcal{H} denoted by $h \otimes a \mapsto h \triangleright a$. Assume also that $\Omega \in \mathcal{H} \otimes \mathcal{H}$ is an invertible element satisfying the cocycle identity

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega).$$

In this case we can consider a new Hopf algebra \mathcal{H}_Ω , defined by Drinfeld [12], such that $\mathcal{H}_\Omega = \mathcal{H}$ as an algebra, but the coproduct is given by $\hat{\Delta}_\Omega = \Omega \hat{\Delta}(\cdot) \Omega^{-1}$. We can also define a new product \star_Ω on $\mathcal{A}_\Omega = \mathcal{A}$ by

$$a \star_\Omega b = m(\Omega^{-1} \triangleright (a \otimes b)),$$

where $m(a \otimes b) = ab$. Then \mathcal{A}_Ω is an \mathcal{H}_Ω -module algebra and, as was observed by Majid [22] (see also [7]), for the corresponding smash, or crossed, products we have

$$\mathcal{H}_\Omega \# \mathcal{A}_\Omega \cong \mathcal{H} \# \mathcal{A}. \quad (0.1)$$

Our goal in this paper is to develop a similar theory in the context of C*-algebras and actions of locally compact quantum groups. Thus, given a C*-algebra A with an action of a locally compact quantum group G on it, and a unitary cocycle $\Omega \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G})$, we want to define a deformation A_Ω of A . Note that the deformed quantum group G_Ω is defined in full generality by the theory developed by De Commer [9].

Particular cases of our construction of A_Ω are of course well-known. For example, when G is dual to a discrete group Γ , $A = \Gamma \rtimes_\gamma B$ (all crossed products in this paper are assumed to be reduced) and the action of G is the dual action $\hat{\gamma}$, then A_Ω is nothing else than the twisted crossed product $\Gamma \rtimes_{\gamma, \Omega} B$, as defined already by Zeller-Meier [39] in the 60s.

In the case when G is a compact group, a study of cocycles on \hat{G} was initiated by Landstad [19] and Wassermann [34, 35] in the early 80s. They defined twisted group C*-algebras $C_r^*(\hat{G}; \Omega)$, which should be thought of as deformations $C_0(G)_\Omega$.

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Another milestone is the work of Rieffel [27] for $G = \mathbb{R}^{2d}$. He was motivated by deformation quantization theory and extended Weyl quantization to actions of \mathbb{R}^{2d} on C^* -algebras. His theory is beautiful, but quite complicated, based on an extension of oscillatory integrals to C^* -algebras. A much simpler, although less explicit, approach was later proposed by Kasprzak [15]. His idea was to start with isomorphism (0.1). This isomorphism implies that A_Ω can be identified with the fixed point subalgebra of $\mathcal{H} \# \mathcal{A}$ with respect to a coaction of \mathcal{H}_Ω that corresponds to the dual coaction on $\mathcal{H}_\Omega \# \mathcal{A}_\Omega$. It is easy to check that this coaction of \mathcal{H}_Ω on $\mathcal{H} \# \mathcal{A}$ is simply the dual coaction of \mathcal{H} twisted by Ω . This allows one to describe A_Ω in terms of $\mathcal{H} \# \mathcal{A}$ and Ω without using expressions like $m(\Omega^{-1} \triangleright (a \otimes b))$ that are difficult to make sense of in the analytic setting. Kasprzak developed this idea in the setting of C^* -algebras in the case when G is an abelian locally compact group and Ω is a continuous 2-cocycle on \hat{G} . But his theory works equally well when G is the dual of a locally compact group.

In the previous paper the first author together with Bhowmick and Sangha [3] extended Kasprzak's approach to the case of measurable cocycles. The problem with such cocycles is that the twisted dual action of \hat{G} on $G \ltimes A$ (since we deal now with group duals the deformed quantum group G_Ω is G) is not always well-defined. Nevertheless, a description of A_Ω in terms of generators continues to make sense and, as was shown in [3], these algebras still satisfy a number of properties to be considered as the correct deformations of A .

A by no means exhaustive list of other relevant papers on cocycle deformations in the operator algebraic setting includes [4], [5], [6], [8], [10], [13], [14], [16], [20], [21], [29], [32], [33], [38].

In this paper we continue the work started in [3] and define the deformations A_Ω for general locally compact quantum groups G and arbitrary unitary cocycles Ω on \hat{G} . In fact, our primary interest is the group case. This is the situation studied by Landstad and Raeburn [20, 21] for C^* -algebras of the form $A = C_0(G/H)$ and a particular class of cocycles on \hat{G} . But since, as follows from the above discussion, the deformed quantum group G_Ω should play a role in the theory, trying to work only with groups and their duals is unnecessarily restrictive. Moreover, in the proofs of a significant number of general results there would be almost no simplifications even if we restricted ourselves to the group situation.

The paper is organized as follows. In Section 1 we collect some basic facts on locally compact quantum groups.

In Section 2 we study various notions related to cocycles. In particular, here we introduce twisted crossed products. They are related, but not in the most straightforward way, to cocycle crossed products studied by Vaes and Vainerman [32]. Another important notion is regularity of a cocycle, which means that the crossed product $C_r^*(\hat{G}; \Omega) \rtimes G$ is isomorphic to the algebra of compact operators on $L^2(G)$. For regular quantum groups we show that regularity of a cocycle Ω is equivalent to the inclusion $(K \otimes 1)\hat{W}\Omega^*(1 \otimes K) \subset K \otimes K$, where K is the algebra of compact operators on $L^2(G)$ and \hat{W} is the multiplicative unitary of \hat{G} . But we leave open the question of finding somewhat more manageable sufficient conditions for regularity.

Section 3 contains our main general results. Here we introduce the deformed algebras A_Ω and study such questions as the relation of A_Ω to twisted crossed products, existence of an action of G_Ω on A_Ω , deformation in stages, invariance of A_Ω under replacing Ω by a cohomologous cocycle.

In Section 4 we specialize to the group case. The main goal is to understand when a cocycle on \hat{G} is regular, but the outcome is far from satisfactory. We observe that Ω is regular if the action of G on $C_r^*(\hat{G}; \Omega)$ is proper and saturated in the sense of Rieffel [26]. What we are able to prove in general is that this action always has a weaker property of integrability; note that integrability in an even weaker sense has already been established by Vaes and Vainerman [32]. One outcome of this discussion is that A_Ω is generated by the image of a dense subspace of $C_r^*(\hat{G}; \Omega) \otimes A$ under the operator-valued weight $C_r^*(\hat{G}; \Omega) \otimes A \rightarrow M(C_r^*(\hat{G}; \Omega) \otimes A)^G$.

In Section 5, in order to illustrate some of our general results, as well as the difficulties one might encounter in analyzing concrete examples, we briefly consider the cocycles on the duals of some

solvable Lie groups recently constructed by Bieliavsky et al. [6, 5]. A detailed study will appear elsewhere.

We finish the paper with a list of open problems in Section 6.

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1. PRELIMINARIES

In this section we will fix our notation and recall some facts on locally compact quantum groups that we will use freely throughout the paper.

1.1. Locally compact quantum groups. Recall [17, 18] that a locally compact quantum group, in the von Neumann algebraic setting, is a pair $G = (M, \Delta)$ consisting of a von Neumann algebra M and a coassociative normal unital $*$ -homomorphism $\Delta: M \rightarrow M \bar{\otimes} M$ such that there exist a left invariant n.s.f weight φ and a right invariant n.s.f. weight ψ on M . We will often use the suggestive notation $L^\infty(G)$ for M . Denote by $L^2(G)$ the space of the GNS-representation of M defined by the left invariant Haar weight φ . Write $\Lambda: \mathcal{N}_\varphi \rightarrow L^2(G)$ for the corresponding map, where $\mathcal{N}_\varphi = \{x \in L^\infty(G) \mid \varphi(x^*x) < \infty\}$. Then the multiplicative unitary W of G is defined by

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Delta(y)(x \otimes 1)), \quad \text{for } x, y \in \mathcal{N}_\varphi.$$

Therefore, identifying $L^\infty(G)$ with its image under the GNS-representation defined by φ , we have

$$\Delta(x) = W^*(1 \otimes x)W \quad \text{for } x \in L^\infty(G).$$

Throughout the whole paper we will denote by K the C*-algebra of compact operators on $L^2(G)$. We identify K^* with $B(L^2(G))_*$. For a subset X of a normed space we denote by $[X]$ the norm closure of the linear span of X . Using this notation the C*-algebra $C_0(G)$ of continuous functions on G vanishing at infinity is defined by

$$C_0(G) = [(\iota \otimes \omega)(W) \mid \omega \in K^*].$$

The dual quantum group $\hat{G} = (\hat{M}, \hat{\Delta})$ is defined by

$$\hat{M} = \{(\omega \otimes \iota)(W) \mid \omega \in K^*\}^{-\sigma\text{-strong}^*}, \quad \hat{\Delta}(x) = \Sigma W(x \otimes 1)W^*\Sigma,$$

where Σ is the flip on $L^2(G) \otimes L^2(G)$. By definition \hat{M} is represented on $L^2(G)$. This representation is identified with the GNS-representation defined by a left invariant Haar weight $\hat{\varphi}$ on \hat{G} , with the corresponding map $\hat{\Lambda}: \mathcal{N}_{\hat{\varphi}} \rightarrow L^2(G)$ uniquely defined by the identities

$$(\hat{\Lambda}((\omega \otimes \iota)(W)), \Lambda(x)) = \omega(x^*)$$

for $x \in \mathcal{N}_\varphi$ and suitable $\omega \in K^*$, namely, for ω such that the map $\Lambda(x) \mapsto \overline{\omega(x^*)}$ extends to a bounded linear functional on $L^2(G)$. Under this identification the multiplicative unitary \hat{W} of \hat{G} is given by

$$\hat{W} = \Sigma W^* \Sigma.$$

We then have

$$C_0(\hat{G}) = [(\omega \otimes \iota)(W) \mid \omega \in K^*] \quad \text{and} \quad W \in M(C_0(G) \otimes C_0(\hat{G})).$$

The pentagon relation for W can be written in the following equivalent forms:

$$(\Delta \otimes \iota)(W) = W_{13}W_{23}, \quad (\iota \otimes \hat{\Delta})(W) = W_{13}W_{12}.$$

Denote by J , resp. \hat{J} , the modular involutions on $L^2(G)$ defined by φ , resp. $\hat{\varphi}$. Then J and \hat{J} commute up to a scalar factor. The unitary antipode on M , resp. \hat{M} , is given by $R(x) = \hat{J}x^*\hat{J}$, resp. $\hat{R}(a) = Ja^*J$. We have $(R \otimes \hat{R})(W) = W$, that is,

$$(\hat{J} \otimes J)W^*(\hat{J} \otimes J) = W.$$

In addition to W it is convenient to use another multiplicative unitary V corresponding to the GNS-representation defined by the right Haar weight ψ . It is defined by

$$V = (\hat{J} \otimes \hat{J})\hat{W}(\hat{J} \otimes \hat{J}) \in L^\infty(\hat{G})' \bar{\otimes} L^\infty(G),$$

and we have

$$\Delta(x) = V(x \otimes 1)V^* \quad \text{for } x \in L^\infty(G).$$

A locally compact quantum group G is called regular [1, 2], if

$$(K \otimes 1)W(1 \otimes K) \subset K \otimes K,$$

or equivalently, $(1 \otimes K)W(K \otimes 1) \subset K \otimes K$. This is equivalent to several other conditions. In particular, if G is regular then $[(K \otimes 1)W(1 \otimes K)] = K \otimes K$ and

$$[(C_0(G) \otimes 1)W(1 \otimes C_0(\hat{G}))] = C_0(G) \otimes C_0(\hat{G}).$$

Recall that genuine locally compact groups, and therefore also their duals, are always regular.

1.2. Actions on operator algebras. A continuous left action of a locally compact quantum group G on a von Neumann algebra N is a normal unital injective $*$ -homomorphism $\alpha: N \rightarrow L^\infty(G) \bar{\otimes} N$ such that $(\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha$. A continuous left action of G on a C^* -algebra A is a non-degenerate injective $*$ -homomorphism $\alpha: A \rightarrow M(C_0(G) \otimes A)$ such that $(\iota \otimes \alpha)\alpha = (\Delta \otimes \iota)\alpha$ and the following cancelation property holds:

$$C_0(G) \otimes A = [(C_0(G) \otimes 1)\alpha(A)].$$

The following proposition is a small variation of results of Baaj, Skandalis and Vaes, see Propositions 5.7 and 5.8 in [2]. We include a complete proof for convenience.

Proposition 1.1. *Assume G is a regular locally compact quantum group, N is a von Neumann algebra and $\alpha: N \rightarrow L^\infty(G) \bar{\otimes} N$ is a continuous action of G on N . For a subspace $X \subset N$ define $X_\alpha = [(\omega \otimes \iota)\alpha(X) \mid \omega \in K^*] \subset N$. Then for any C^* -subalgebra $A \subset N$, if $A_\alpha \subset A$, then A_α is a C^* -algebra and $\alpha|_{A_\alpha}$ is a continuous action of G on A_α .*

Proof. Since $AA_\alpha \subset A$ by assumption, we have

$$\begin{aligned} A_\alpha &\supset [(\omega \otimes \iota)\alpha(A(\nu \otimes \iota)\alpha(A)) \mid \omega, \nu \in K^*] \\ &= [(\nu \otimes \omega \otimes \iota)(\alpha(A)_{23}(\Delta \otimes \iota)\alpha(A)) \mid \omega, \nu \in K^*] \\ &= [(\nu \otimes \omega \otimes \iota)(\alpha(A)_{23}V_{12}\alpha(A)_{13}V_{12}^*) \mid \omega, \nu \in K^*] \\ &= [(\nu \otimes \omega \otimes \iota)(\alpha(A)_{23}V_{12}\alpha(A)_{13}) \mid \omega, \nu \in K^*] \\ &= [(\nu \otimes \omega \otimes \iota)(\alpha(A)_{23}\alpha(A)_{13}) \mid \omega, \nu \in K^*], \end{aligned}$$

where in the last step we used that $[(K \otimes 1)V(1 \otimes K)] = K \otimes K$ by regularity. We thus see that $A_\alpha A_\alpha \subset A_\alpha$. It is also clear that A_α is invariant under the $*$ -operation. Thus A_α is a C^* -algebra.

In order to show that $\alpha|_{A_\alpha}$ is an action, observe first that $(X_\alpha)_\alpha = X_\alpha$ for any subspace $X \subset N$. Therefore replacing A by A_α we may assume that $A = A_\alpha$. We then have

$$\begin{aligned} [\alpha(A)(C_0(G) \otimes 1)] &= [\alpha((\omega \otimes \iota)\alpha(A))(C_0(G) \otimes 1) \mid \omega \in K^*] \\ &= [(\omega \otimes \iota \otimes \iota)(V_{12}\alpha(A)_{13}V_{12}^*(1 \otimes C_0(G) \otimes 1)) \mid \omega \in K^*] \\ &= [(\omega \otimes \iota \otimes \iota)(V_{12}\alpha(A)_{13}(1 \otimes C_0(G) \otimes 1)) \mid \omega \in K^*] \\ &= [(\omega \otimes \iota \otimes \iota)(\alpha(A)_{13}(1 \otimes C_0(G) \otimes 1)) \mid \omega \in K^*], \end{aligned}$$

where in the last step we used that $[(K \otimes 1)V(1 \otimes C_0(G))] = K \otimes C_0(G)$ by regularity. Therefore

$$[\alpha(A)(C_0(G) \otimes 1)] = C_0(G) \otimes A.$$

From this we conclude that $\alpha(A) \subset M(C_0(G) \otimes A)$ and $\alpha|_A$ is a continuous action of G . \square

1.3. Crossed products and duality. Given a continuous left action α of a locally compact quantum group G on a C*-algebra A , the reduced C*-crossed product $G \ltimes_\alpha A$ (since we are going to consider only reduced crossed products, we omit r in the notation) is defined by

$$G \ltimes_\alpha A = [(C_0(\hat{G}) \otimes 1)\alpha(A)] \subset M(K \otimes A).$$

It is equipped with the dual continuous right action of \hat{G} , or in other words, with a continuous left action $\hat{\alpha}$ of \hat{G}^{op} , which is the quantum group \hat{G} with the opposite comultiplication $\hat{\Delta}^{\text{op}}$ on $L^\infty(\hat{G}^{\text{op}}) = L^\infty(\hat{G})$. Namely, we have

$$\hat{\alpha}(x) = (\hat{W}^{\text{op}} \otimes 1)^*(1 \otimes x)(\hat{W}^{\text{op}} \otimes 1) \quad \text{for } x \in G \ltimes_\alpha A,$$

where

$$\hat{W}^{\text{op}} = (J \otimes J)\hat{W}(J \otimes J) \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(G)'$$

is the multiplicative unitary of \hat{G}^{op} (see [18, Section 4]), so

$$\hat{\alpha}(\alpha(a)) = 1 \otimes \alpha(a) \quad \text{for } a \in A \quad \text{and} \quad \hat{\alpha}(x \otimes 1) = \hat{\Delta}^{\text{op}}(x) \otimes 1 \quad \text{for } x \in C_0(\hat{G}).$$

Then the double crossed product is

$$\hat{G}^{\text{op}} \ltimes_{\hat{\alpha}} G \ltimes_\alpha A = [(JC_0(G)J \otimes 1 \otimes 1)(\hat{\Delta}^{\text{op}}(C_0(\hat{G})) \otimes 1)(1 \otimes \alpha(A))].$$

Since

$$\hat{\Delta}^{\text{op}}(x) = W(x \otimes 1)W^* \quad \text{for } x \in C_0(\hat{G}),$$

the map $\text{Ad}(W^* \otimes 1)$ maps $\hat{G}^{\text{op}} \ltimes_{\hat{\alpha}} G \ltimes_\alpha A$ onto

$$[(JC_0(G)JC_0(\hat{G}) \otimes 1 \otimes 1)(\iota \otimes \alpha)\alpha(A)].$$

In particular, if $[JC_0(G)JC_0(\hat{G})] = K$, which is another equivalent formulation of regularity of G , we get the Takesaki-Takai duality

$$\hat{G}^{\text{op}} \ltimes_{\hat{\alpha}} G \ltimes_\alpha A \cong K \otimes \alpha(A) \cong K \otimes A.$$

Assume now that G is regular and β is a continuous left action of \hat{G}^{op} on a C*-algebra B . Assume also that there exists a unitary X in $M(C_0(G) \otimes B)$ such that

$$(\Delta \otimes \iota)(X) = X_{13}X_{23} \quad \text{and} \quad (\iota \otimes \beta)(X) = W_{12}X_{13}.$$

Consider the *-homomorphism

$$\eta: B \rightarrow M(K \otimes B), \quad \eta(x) = X^*\beta(x)X.$$

Then by a Landstad-type result of Vaes [31, Theorem 6.7], the space

$$A = [(\omega \otimes \iota)\eta(B) \mid \omega \in K^*] \subset M(B)$$

is a C*-algebra, the formula

$$\alpha(a) = X^*(1 \otimes a)X$$

defines a continuous left action of G on A , and η defines an isomorphism $B \cong G \ltimes_\alpha A$ intertwining β with $\hat{\alpha}$. Note that if we already have $(B, \beta) = (G \ltimes_\alpha A, \hat{\alpha})$, then we can take $X = W \otimes 1$, in which case η becomes the map defining the Takesaki-Takai isomorphism.

2. DUAL COCYCLES

2.1. Twisted group algebras. Assume G is a locally compact quantum group. By a measurable unitary dual 2-cocycle on G , or a measurable unitary 2-cocycle on \hat{G} , we mean a unitary element $\Omega \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G})$ such that

$$(\Omega \otimes 1)(\hat{\Delta} \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \hat{\Delta})(\Omega).$$

We say that Ω is continuous if $\Omega \in M(C_0(\hat{G}) \otimes C_0(\hat{G}))$.

Given a measurable unitary 2-cocycle Ω , the cocycle condition can be written as

$$(\hat{\Delta} \otimes \iota)(\hat{W}\Omega^*)_{12} = (\hat{W}\Omega^*)_{13}(\hat{W}\Omega^*)_{23}. \quad (2.1)$$

Indeed, we have

$$\begin{aligned} (\hat{\Delta} \otimes \iota)(\hat{W}\Omega^*) &= \hat{W}_{13}\hat{W}_{23}(\hat{\Delta} \otimes \iota)(\Omega^*) \\ &= \hat{W}_{13}\hat{W}_{23}(\iota \otimes \hat{\Delta})(\Omega^*)(1 \otimes \Omega^*)(\Omega \otimes 1) \\ &= \hat{W}_{13}\hat{W}_{23}\hat{W}_{23}^*\Omega_{13}^*\hat{W}_{23}\Omega_{23}^*\Omega_{12}, \end{aligned}$$

which is what we claimed.

Identity (2.1) shows that the space of operators $(\omega \otimes \iota)(\hat{W}\Omega^*)$, $\omega \in K^*$, forms an algebra. The C^* -algebra $C_r^*(\hat{G}; \Omega)$ generated by this algebra is called the reduced twisted group C^* -algebra of \hat{G} . The von Neumann algebra $C_r^*(\hat{G}; \Omega)'' \subset B(L^2(G))$ is denoted by $W^*(\hat{G}; \Omega)$.

Proposition 2.1. *We have $C_r^*(\hat{G}; \Omega) = [(\omega \otimes \iota)(\hat{W}\Omega^*) \mid \omega \in K^*]$.*

When G is a compact quantum group, this was proved in [4, Lemma 4.9]. The argument works for any regular locally compact quantum group. The general case will be treated in Section 3.1 below by constructing a different set of generators of $C_r^*(\hat{G}; \Omega)$.

Definition 2.2. We say that a cocycle Ω is *admissible* if $\hat{W}\Omega^* \in M(K \otimes C_r^*(\hat{G}; \Omega))$.

Since the multiplicative unitary W of a locally compact quantum group G is always manageable [18, 36], it is plausible that any cocycle is admissible. We at least have the following result.

Proposition 2.3. *If G is regular, then any measurable unitary 2-cocycle on \hat{G} is admissible.*

Proof. We adapt the proof of [1, Proposition 3.6] of a similar result for the multiplicative unitary. Rewrite identity (2.1) as

$$\hat{W}_{12}^*(\hat{W}\Omega^*)_{23}(\hat{W}\Omega^*)_{12} = (\hat{W}\Omega^*)_{13}(\hat{W}\Omega^*)_{23}. \quad (2.2)$$

Multiplying by $K \otimes 1 \otimes 1$ on the right and applying the slice maps to the second leg we get

$$[(\iota \otimes \omega \otimes \iota)(\hat{W}_{12}^*(\hat{W}\Omega^*)_{23}(K \otimes 1 \otimes 1)) \mid \omega \in K^*] = \hat{W}\Omega^*(K \otimes C_r^*(\hat{G}; \Omega)).$$

Using that $[(1 \otimes K)\hat{W}^*(K \otimes 1)] = K \otimes K$ by regularity, we see that the left hand side equals $K \otimes C_r^*(\hat{G}; \Omega)$, so

$$\hat{W}\Omega^*(K \otimes C_r^*(\hat{G}; \Omega)) = K \otimes C_r^*(\hat{G}; \Omega).$$

Similarly, rewriting (2.2) as

$$(\hat{W}\Omega^*)_{12}(\hat{W}\Omega^*)_{23}^* = (\hat{W}\Omega^*)_{23}^*\hat{W}_{12}(\hat{W}\Omega^*)_{13},$$

multiplying this identity by $K \otimes 1 \otimes 1$ on the left and applying the slice maps to the second leg, we get

$$K \otimes C_r^*(\hat{G}; \Omega) = (K \otimes C_r^*(\hat{G}; \Omega))\hat{W}\Omega^*.$$

Therefore $\hat{W}\Omega^* \in M(K \otimes C_r^*(\hat{G}; \Omega))$. □

Let us also note the following.

Proposition 2.4. *If Ω is an admissible continuous unitary 2-cocycle on \hat{G} , then*

$$\hat{W}\Omega^* \in M(C_0(\hat{G}) \otimes C_r^*(G; \Omega)).$$

Proof. By admissibility we have $\hat{W}\Omega^* \in M(K \otimes C_r^*(\hat{G}; \Omega))$. Using identity (2.2) in the form

$$(\hat{W}\Omega^*)_{13} = \hat{W}_{12}^*(\hat{W}\Omega^*)_{23}\hat{W}_{12}\Omega_{12}^*(\hat{W}\Omega^*)_{23}^*,$$

we see that $(\hat{W}\Omega^*)_{13} \in M(C_0(\hat{G}) \otimes K \otimes C_r^*(\hat{G}; \Omega))$. Applying the slice maps to the second leg we conclude that $\hat{W}\Omega^* \in M(C_0(\hat{G}) \otimes C_r^*(G; \Omega))$. \square

The von Neumann algebras $W^*(\hat{G}; \Omega)$ (in fact, more general von Neumann-algebraic cocycle crossed products) were extensively studied by Vaes and Vainerman [32]. In particular, in [32, Proposition 1.4] they showed that there exists a continuous right action β of G on $W^*(\hat{G}; \Omega)$ such that

$$(\iota \otimes \beta)(\hat{W}\Omega^*) = \hat{W}_{13}(\hat{W}\Omega^*)_{12}.$$

This action is given by

$$\beta(x) = V(x \otimes 1)V^* \quad \text{for } x \in W^*(\hat{G}; \Omega).$$

Another useful formula, which follows from (2.2), see [32, Proposition 1.5], is

$$\beta(x) = (\hat{W}\Omega^*)_{21}(1 \otimes x)(\hat{W}\Omega^*)_{21}^*. \quad (2.3)$$

Proposition 2.5. *The restriction of β to $C_r^*(\hat{G}; \Omega)$ defines a continuous action of G on the C^* -algebra $C_r^*(\hat{G}; \Omega)$.*

This is easy to prove for admissible cocycles, in particular, for regular locally compact quantum groups. As with Proposition 2.1, the general case will be treated in Section 3.1 below.

2.2. Deformed quantum group. Given a unitary 2-cocycle $\Omega \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G})$, we can define a new coproduct $\hat{\Delta}_\Omega$ on $L^\infty(\hat{G})$ by

$$\hat{\Delta}_\Omega(x) = \Omega \hat{\Delta}(x) \Omega^* \quad \text{for } x \in L^\infty(\hat{G}).$$

By a result of De Commer [9], the pair $\hat{G}_\Omega = (L^\infty(\hat{G}), \hat{\Delta}_\Omega)$ is again a locally compact quantum group. We will use the subscript Ω to denote the objects related to \hat{G}_Ω , such as the coproduct, the multiplicative unitary, etc.

In order to describe the multiplicative unitary \hat{W}_Ω of \hat{G}_Ω we need to recall some results of Vaes and Vainerman [32]. By [32, Lemma 1.12] the action β of G on $W^*(\hat{G}; \Omega)$ is integrable, meaning that $(\iota \otimes \varphi)\beta$ is a n.s.f. operator valued weight from $W^*(\hat{G}; \Omega)$ to $W^*(\hat{G}; \Omega)^\beta = \mathbb{C}1$. Therefore we have a n.s.f. weight $\tilde{\varphi}$ on $W^*(\hat{G}; \Omega)$ such that

$$\tilde{\varphi}(x)1 = (\iota \otimes \varphi)\beta(x) \quad \text{for } x \in W^*(\hat{G}; \Omega)_+.$$

By construction $W^*(\hat{G}; \Omega)$ is represented on $L^2(G)$. By [32, Proposition 1.15] this representation can be identified with the GNS-representation defined by the weight $\tilde{\varphi}$, with the corresponding map $\tilde{\Lambda}: \mathcal{N}_{\tilde{\varphi}} \rightarrow L^2(G)$ given by

$$\tilde{\Lambda}((\omega \otimes \iota)(\hat{W}\Omega^*)) = \Lambda((\omega \otimes \iota)(\hat{W}))$$

for suitable $\omega \in K^*$. Denote by \tilde{J} the modular involution on $L^2(G)$ defined by $\tilde{\varphi}$. The von Neumann algebra $L^\infty(\hat{G}_\Omega) = L^\infty(\hat{G}) \subset B(L^2(G))$ is in the standard form, so $\hat{J}_\Omega = \hat{J}$, and by [9, Proposition 5.4] we have

$$\hat{W}_\Omega = (\tilde{J} \otimes \hat{J})\Omega\hat{W}^*(J \otimes \hat{J})\Omega^*. \quad (2.4)$$

From this we immediately get the following proposition.

Proposition 2.6. *For any measurable unitary 2-cocycle $\Omega \in L^\infty(\hat{G}) \bar{\otimes} L^\infty(\hat{G})$ on \hat{G} , the element Ω^* is a measurable unitary 2-cocycle on \hat{G}_Ω , and we have*

$$C_r^*(\hat{G}_\Omega; \Omega^*) = \hat{J} C_r^*(\hat{G}; \Omega) \hat{J}.$$

If, furthermore, the cocycle Ω on \hat{G} is admissible, then the cocycle Ω^ on \hat{G}_Ω is also admissible.*

Therefore $C_r^*(\hat{G}; \Omega)$ is $*$ -anti-isomorphic to $C_r^*(\hat{G}_\Omega; \Omega^*)$. By Proposition 2.5, which we yet have to prove, we have a continuous right action of G_Ω on $C_r^*(\hat{G}_\Omega; \Omega^*)$, where G_Ω is the dual of \hat{G}_Ω . Using the unitary antipode $R_\Omega(x) = \hat{J}_\Omega x^* \hat{J}_\Omega = \hat{J} x^* \hat{J}$ on $C_0(G_\Omega)$ we can transform this action to a continuous left action β_Ω of G_Ω on $C_r^*(\hat{G}; \Omega)$.

Lemma 2.7. *We have*

$$\beta_\Omega(x) = W_\Omega^*(1 \otimes x) W_\Omega \quad \text{for } x \in C_r^*(\hat{G}; \Omega),$$

and

$$(\iota \otimes \beta_\Omega)(\hat{W}\Omega^*) = (\hat{W}\Omega^*)_{13}((\tilde{J} \otimes \hat{J})\hat{W}_\Omega^*(\tilde{J} \otimes \hat{J}))_{12}.$$

Proof. The right action of G_Ω on $C_r^*(\hat{G}_\Omega; \Omega^*)$ is given by $x \mapsto V_\Omega(x \otimes 1)V_\Omega$. Therefore the left action of G_Ω on $C_r^*(\hat{G}; \Omega)$ is defined by

$$\beta_\Omega(x) = (\hat{J} \otimes \hat{J})(V_\Omega(\hat{J}x\hat{J} \otimes 1)V_\Omega)_{21}(\hat{J} \otimes \hat{J}).$$

Since

$$(V_\Omega)_{21} = (\hat{J} \otimes \hat{J})(\hat{W}_\Omega)_{21}(\hat{J} \otimes \hat{J}) = (\hat{J} \otimes \hat{J})W_\Omega^*(\hat{J} \otimes \hat{J}),$$

we get the first formula for β_Ω in the formulation.

Similarly, since the right action of G_Ω on $C_r^*(\hat{G}_\Omega; \Omega^*)$ maps $(\omega \otimes \iota)(\hat{W}_\Omega\Omega)$ into

$$(\omega \otimes \iota \otimes \iota)((\hat{W}_\Omega)_{13}(\hat{W}_\Omega\Omega)_{12}),$$

we have

$$(\omega \otimes \beta_\Omega)((J_1 \otimes \hat{J})\hat{W}_\Omega\Omega(J_2 \otimes \hat{J})) = (\omega \otimes \iota \otimes \iota)((J_1 \otimes \hat{J} \otimes \hat{J})(\hat{W}_\Omega)_{12}(\hat{W}_\Omega\Omega)_{13}(J_2 \otimes \hat{J} \otimes \hat{J}))$$

for any $\omega \in K^*$ and any bounded antilinear operators J_1 and J_2 . Since

$$\hat{W}_\Omega\Omega = (\tilde{J} \otimes \hat{J})(\hat{W}\Omega^*)^*(J \otimes \hat{J}),$$

taking $J_1 = \tilde{J}$ and $J_2 = J$ we get

$$(\iota \otimes \beta_\Omega)((\hat{W}\Omega^*)^*) = ((\tilde{J} \otimes \hat{J})\hat{W}_\Omega(\tilde{J} \otimes \hat{J}))_{12}(\hat{W}\Omega^*)_{13}^*,$$

which is exactly the second formula in the formulation. \square

Therefore we have a left action β_Ω of G_Ω and a right action β of G on $C_r^*(\hat{G}; \Omega)$. Using that $(\iota \otimes \beta)(\hat{W}\Omega^*) = \hat{W}_{13}(\hat{W}\Omega^*)_{12}$ and the second formula in the lemma above, we see that these actions commute: $(\beta_\Omega \otimes \iota)\beta = (\iota \otimes \beta)\beta_\Omega$.

2.3. Twisted crossed products. Assume Ω is a measurable unitary 2-cocycle on \hat{G} and α is a continuous left action of \hat{G}^{op} on a C^* -algebra A .

Definition 2.8. The *reduced twisted crossed product* $\hat{G}^{\text{op}} \ltimes_{\alpha, \Omega} A$ is defined as the C^* -subalgebra of $M(K \otimes A)$ generated by $(J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)\alpha(A)$.

Proposition 2.9. *We have $\hat{G}^{\text{op}} \ltimes_{\alpha, \Omega} A = [(J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)\alpha(A)]$, and the formula*

$$\hat{\alpha}(x) = \text{Ad}((1 \otimes J\hat{J} \otimes 1)(W_\Omega^* \otimes 1)(1 \otimes \hat{J}J \otimes 1))(1 \otimes x)$$

defines a continuous left action of G_Ω on $\hat{G}^{\text{op}} \ltimes_{\alpha, \Omega} A$.

Proof. The first part is proved in the standard way. Namely, observe first that

$$\hat{W}^{\text{op}} = (J \otimes J)\hat{W}(J \otimes J) = (1 \otimes J\hat{J})\hat{W}^*(1 \otimes \hat{J}J),$$

whence

$$\hat{\Delta}^{\text{op}}(x) = \text{Ad}((1 \otimes J\hat{J})\hat{W}\Omega^*(1 \otimes \hat{J}J))(1 \otimes x) \quad \text{for } x \in L^\infty(\hat{G}).$$

It follows that

$$\begin{aligned} (1 \otimes J\hat{J} \otimes 1)(\hat{W}\Omega^* \otimes 1)(1 \otimes \hat{J}J \otimes 1)(1 \otimes \alpha(A)) \\ = (\iota \otimes \alpha)\alpha(A)(1 \otimes J\hat{J} \otimes 1)(\hat{W}\Omega^* \otimes 1)(1 \otimes \hat{J}J \otimes 1). \end{aligned}$$

Applying the slice maps to the first leg and using that $(K \otimes 1)\alpha(A) \subset K \otimes A$ we conclude that

$$[(J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)\alpha(A)] \subset [\alpha(A)(J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)],$$

and therefore

$$\hat{G}^{\text{op}} \ltimes_{\alpha, \Omega} A = [\alpha(A)(J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)] = [(J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)\alpha(A)].$$

For the second part, note that the element

$$(1 \otimes J\hat{J} \otimes 1)(W_\Omega^* \otimes 1)(1 \otimes \hat{J}J \otimes 1) \in L^\infty(G_\Omega) \bar{\otimes} L^\infty(\hat{G})' \otimes 1$$

commutes with $1 \otimes \alpha(A)$. Therefore it suffices to check that the formula

$$J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \ni x \mapsto \text{Ad}((1 \otimes J\hat{J})W_\Omega^*(1 \otimes \hat{J}J))(1 \otimes x)$$

defines a continuous action of G_Ω on $J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J$. But this is true by Lemma 2.7. \square

Twisted crossed products, or cocycle crossed products, in the von Neumann algebraic setting were defined by Vaes and Vainerman [32]. Our definition is of course related to theirs, but not in the most straightforward way. Namely, assume we are given a continuous left action α of \hat{G}^{op} on a von Neumann algebra N . Then (α, Ω_{21}^*) is a cocycle action of $(\hat{G}^{\text{op}})_{\Omega_{21}} = (\hat{G}_\Omega)^{\text{op}}$ on N in the sense of [32, Definition 1.1]. The von Neumann-algebraic cocycle crossed product of N by $(\hat{G}^{\text{op}})_{\Omega_{21}}$ is defined as the von Neumann algebra generated by $\alpha(N)$ and $W^*((\hat{G}^{\text{op}})_{\Omega_{21}}; \Omega_{21}^*) \otimes 1$, see [32, Definition 1.3].

Lemma 2.10. *Letting $X = \tilde{J}J$, we have $X \in L^\infty(\hat{G})$ and*

$$J\hat{J}W^*(\hat{G}; \Omega)\hat{J}J = \hat{J}X^*\hat{J}W^*((\hat{G}^{\text{op}})_{\Omega_{21}}; \Omega_{21}^*)\hat{J}X\hat{J}.$$

Proof. The claim that $X \in L^\infty(\hat{G})$ is in [9, Section 5]. By Proposition 2.6, applied to the cocycle Ω_{21} on \hat{G}^{op} , we have

$$W^*((\hat{G}^{\text{op}})_{\Omega_{21}}; \Omega_{21}^*) = \hat{J}W^*(\hat{G}^{\text{op}}; \Omega_{21})\hat{J}.$$

By [9, Proposition 5.3] we also have

$$\Omega^*(X \otimes X) = \hat{\Delta}(X)(\hat{R} \otimes \hat{R})(\Omega_{21}) = \hat{\Delta}(X)(J \otimes J)\Omega_{21}^*(J \otimes J),$$

whence

$$\hat{W}\Omega^*(X \otimes X) = (1 \otimes X)\hat{W}(J \otimes J)\Omega_{21}^*(J \otimes J) = (1 \otimes X)(J \otimes J)\hat{W}^{\text{op}}\Omega_{21}^*(J \otimes J),$$

and therefore

$$W^*(\hat{G}; \Omega) = XJW^*(\hat{G}^{\text{op}}; \Omega_{21})JX^* = \tilde{J}W^*(\hat{G}^{\text{op}}; \Omega_{21})\tilde{J}.$$

It follows that

$$J\hat{J}W^*(\hat{G}; \Omega)\hat{J}J = \hat{J}J\tilde{J}W^*(\hat{G}^{\text{op}}; \Omega_{21})\tilde{J}J\hat{J} = \hat{J}J\tilde{J}\hat{J}W^*((\hat{G}^{\text{op}})_{\Omega_{21}}; \Omega_{21}^*)\hat{J}\tilde{J}J\hat{J},$$

which is what we need. \square

Therefore up to conjugation by $\hat{J}X^*\hat{J} \otimes 1$ our definition of the twisted crossed product by \hat{G}^{op} is a C*-algebraic version of the definition of Vaes and Vainerman of the cocycle crossed product by $(\hat{G}^{\text{op}})_{\Omega_{21}}$.

2.4. Regular cocycles. Assume Ω is a measurable unitary cocycle on \hat{G} . Recall that we have a continuous right action β of G on $C_r^*(\hat{G}; \Omega)$, so we can consider the reduced crossed product

$$C_r^*(\hat{G}; \Omega) \rtimes_{\beta} G = [\beta(C_r^*(\hat{G}; \Omega))(1 \otimes \hat{J}C_0(\hat{G})\hat{J})].$$

By (2.3) we have $\beta(x) = (\hat{W}\Omega^*)_{21}(1 \otimes x)(\hat{W}\Omega^*)_{21}^*$. The unitary $(\hat{W}\Omega^*)_{21}^*$ commutes with $1 \otimes \hat{J}C_0(\hat{G})\hat{J}$. Therefore the conjugation by this unitary maps $C_r^*(\hat{G}; \Omega) \rtimes_{\beta} G$ onto $1 \otimes [C_r^*(\hat{G}; \Omega)\hat{J}C_0(\hat{G})\hat{J}]$.

Definition 2.11. A cocycle Ω is called *regular* if $[C_r^*(\hat{G}; \Omega)\hat{J}C_0(\hat{G})\hat{J}] = K$.

Note that by a version of the Takesaki duality [32, Proposition 1.20] the von Neumann algebra generated by $C_r^*(\hat{G}; \Omega)\hat{J}C_0(\hat{G})\hat{J}$ coincides with $B(L^2(G))$ (this will also become clear from the proof of Proposition 2.13 below). Therefore regularity of Ω is equivalent to the formally weaker condition

$$C_r^*(\hat{G}; \Omega)\hat{J}C_0(\hat{G})\hat{J} \subset K.$$

Since the representation of $C_r^*(\hat{G}; \Omega) \rtimes_{\beta} G$ on $L^2(G)$ is faithful and irreducible, yet another equivalent formulation of regularity of Ω is that the C^* -algebra $C_r^*(\hat{G}; \Omega) \rtimes_{\beta} G$ is isomorphic to the algebra of compact operators on some Hilbert space.

By definition, regularity of the trivial cocycle 1 is the same as regularity of G . Therefore regularity of cocycles is definitely not automatic. Even for regular locally compact quantum groups regularity of a cocycle is a very delicate question. The only easy cases seem to be covered by the following proposition.

Proposition 2.12. *Any measurable unitary 2-cocycle on \hat{G} is regular in the following cases:*

- (i) \hat{G} is a genuine locally compact group;
- (ii) \hat{G} is a discrete quantum group.

Proof. Part (i) is well-known and is proved in the same way as regularity of \hat{G} , by observing that the space $C_r^*(\hat{G}; \Omega)\hat{J}C_0(\hat{G})\hat{J}$ contains a lot of integral operators. Part (ii) is obvious, as already the algebra $C_0(\hat{G})$ consists of compact operators. \square

In view of various equivalent characterizations of regularity of quantum groups, it is natural to wonder how regularity of a cocycle is related to properties like $(K \otimes 1)\hat{W}\Omega^*(1 \otimes K) \subset K \otimes K$. We have the following result.

Proposition 2.13. *For a cocycle Ω on \hat{G} consider the following conditions:*

- (i) Ω is regular;
- (ii) $(K \otimes 1)\hat{W}\Omega^*(1 \otimes K) \subset K \otimes K$.

Then (i) \Rightarrow (ii). If G is regular, then the two conditions are equivalent.

Proof. In this proof it will be convenient to consider the right action β of G on $C_r^*(\hat{G}; \Omega)$ as the left action β' of G^{op} , so $\beta'(x) = V_{21}(1 \otimes x)V_{21}^*$ for $x \in C_r^*(\hat{G}; \Omega)$. On the von Neumann algebra level, up to stabilization this action is dual. Namely, by [32, Propositions 1.8 and 1.9] the unitary $Y = \hat{V}_{21}^*\Omega_{21}^*$ defines a continuous left action γ of \hat{G} on $B(L^2(G))$ by

$$\gamma(x) = Y(1 \otimes x)Y^*,$$

and we have an isomorphism

$$W^*(\hat{G}; \Omega) \bar{\otimes} B(L^2(G)) \cong (\hat{J}L^\infty(\hat{G})\hat{J} \otimes 1 \cup \gamma(B(L^2(G))))'', \quad x \mapsto YxY^*,$$

intertwining $\beta' \otimes \iota$ with the dual action $\hat{\gamma}$, defined by $\hat{\gamma}(x) = V_{21}(1 \otimes x)V_{21}^*$. Note also that

$$\hat{V}_{21}^* = (J \otimes J)\hat{W}(J \otimes J) = \hat{W}^{\text{op}}$$

and, as was already used in the proof of Lemma 2.10, that

$$\hat{W}\Omega^*(\tilde{J} \otimes \tilde{J}) = (J \otimes \tilde{J})\hat{W}^{\text{op}}\Omega_{21}^*,$$

so that

$$Y = \hat{V}_{21}^* \Omega_{21}^* = (J \otimes \tilde{J}) \hat{W} \Omega^* (\tilde{J} \otimes \tilde{J}).$$

We are now ready to prove the proposition.

Assume condition (i) holds. We claim that the restriction of γ to K defines a continuous action of \hat{G} on K . Since $K = [C_r^*(\hat{G}; \Omega) \hat{J} C_0(\hat{G}) \hat{J}]$ and Y commutes with $1 \otimes C_r^*(\hat{G}; \Omega)$, it suffices to show that the restriction of γ to $\hat{J} C_0(\hat{G}) \hat{J}$ defines a continuous action. But this is clear, since Ω_{21} commutes with $1 \otimes \hat{J} C_0(\hat{G}) \hat{J}$ and therefore

$$\gamma(\hat{J} x \hat{J}) = \hat{W}^{\text{op}}(1 \otimes \hat{J} x \hat{J})(\hat{W}^{\text{op}})^* = (J \otimes \hat{J})(\hat{W}^{\text{op}})^*(1 \otimes x) \hat{W}^{\text{op}}(J \otimes \hat{J}) = (J \otimes \hat{J}) \hat{\Delta}^{\text{op}}(x)(J \otimes \hat{J}).$$

It follows that

$$(K \otimes 1)Y(1 \otimes K)Y^*(K \otimes 1) = (K \otimes 1)\gamma(K)(K \otimes 1) \subset K \otimes K.$$

This means exactly that $(K \otimes 1)Y(1 \otimes K) \subset K \otimes K$, which is equivalent to (ii).

Assume now that G is regular and condition (ii) holds. We claim again that the restriction of γ to K defines a continuous action of \hat{G} . By Proposition 1.1 it suffices to show that $K_\gamma = K$. Condition (ii) implies that $(K \otimes 1)\gamma(K)(K \otimes 1) \subset K \otimes K$, whence $K_\gamma \subset K$. By Proposition 1.1 this already shows that K_γ is a C*-algebra. Since it is σ -strongly* dense in

$$[(\omega \otimes \iota)\gamma(B(L^2(G))) \mid \omega \in K^*]'' = B(L^2(G)),$$

it follows that $K_\gamma = K$.

Next, we have an isomorphism

$$C_r^*(\hat{G}; \Omega) \otimes K \cong \hat{G} \ltimes_\gamma K, \quad x \mapsto YxY^*,$$

intertwining $\beta' \otimes \iota$ with $\hat{\gamma}$. This is a C*-algebraic version of [32, Propositions 1.8], and the proof is basically the same. Briefly, we have the identity

$$Y_{23}^* \hat{W}_{12} Y_{23} = (\hat{W} \Omega^*)_{12} Y_{13}^*,$$

which is proved similarly to (2.1). Applying the slice maps to the first leg we conclude that $\text{Ad } Y^*$ maps

$$\hat{G} \ltimes_\gamma K = [(C_0(G) \otimes 1)\gamma(K)] = [(\omega \otimes \iota \otimes \iota)(\hat{W} \otimes 1)(1 \otimes Y)(1 \otimes 1 \otimes K)(1 \otimes Y^*) \mid \omega \in K^*]$$

onto

$$[(\omega \otimes \iota \otimes \iota)(\hat{W} \Omega^* \otimes 1)Y_{13}^*(1 \otimes 1 \otimes K) \mid \omega \in K^*] = C_r^*(\hat{G}; \Omega) \otimes K,$$

as claimed.

Consider now the double crossed product $G^{\text{op}} \ltimes_{\hat{\gamma}} \hat{G} \ltimes_\gamma K$. By the Takesaki-Takai duality it is isomorphic to $K \otimes K$. What is however important to us is only the equality

$$[(\hat{J} C_0(\hat{G}) \hat{J} \otimes 1)(\hat{G} \ltimes_\gamma K)] = K \otimes K,$$

which is an immediate consequence of regularity of G . Applying $\text{Ad } Y^*$ to both sides and using that Y commutes with $\hat{J} C_0(\hat{G}) \hat{J} \otimes 1$ we conclude that

$$[\hat{J} C_0(\hat{G}) \hat{J} C_r^*(\hat{G}; \Omega)] \otimes K = K \otimes K,$$

so Ω is regular. □

3. DEFORMATION OF C^* -ALGEBRAS

3.1. Ω -Deformation. Assume A is a C^* -algebra and α is a continuous left action of a locally compact quantum group G on A . Assume Ω is a measurable unitary 2-cocycle on \hat{G} . Consider the crossed product $G \ltimes_\alpha A$ and the dual action $\hat{\alpha}$ of \hat{G}^{op} on $G \ltimes_\alpha A$. We can then try to define a new deformed action $\hat{\alpha}_\Omega$ of $(\hat{G}_\Omega)^{\text{op}}$ on $G \ltimes_\alpha A$ by

$$\hat{\alpha}_\Omega(x) = \Omega_{21} \hat{\alpha}(x) \Omega_{21}^*.$$

Consider also the unitary $W_\Omega \otimes 1$. If $\hat{\alpha}_\Omega$ is well-defined, G_Ω is regular and

$$W_\Omega \otimes 1 \in M(C_0(G_\Omega) \otimes (G \ltimes_\alpha A)),$$

then, as we discussed in Section 1.3, the action $\hat{\alpha}_\Omega$ is dual to an action α_Ω of G_Ω on some C^* -subalgebra of $M(G \ltimes_\alpha A)$ that can be recovered using the homomorphism

$$\eta_\Omega: G \ltimes_\alpha A \rightarrow M(K \otimes K \otimes A), \quad \eta_\Omega(x) = (W_\Omega)_{12}^* \hat{\alpha}_\Omega(x) (W_\Omega)_{12}.$$

We can write this homomorphism as

$$\eta_\Omega(x) = (W_\Omega)_{12}^* \Omega_{21} \hat{\alpha}(x) \Omega_{21}^* (W_\Omega)_{12} = (\hat{W}_\Omega \Omega)_{21} \hat{\alpha}(x) (\hat{W}_\Omega \Omega)_{21}^*.$$

In this form it is defined without any assumptions. This motivates the following definition.

Definition 3.1. The Ω -deformation A_Ω of a G - C^* -algebra A is the C^* -subalgebra of $M(K \otimes A)$ generated by elements of the form $(\omega \otimes \iota \otimes \iota) \eta_\Omega(x)$ for $x \in G \ltimes_\alpha A$ and $\omega \in K^*$.

Note that for $x = y \otimes 1 \in C_0(\hat{G}) \otimes 1 \in M(G \ltimes_\alpha A)$ we have

$$\eta_\Omega(x) = (\hat{W}_\Omega)_{21} (\hat{\Delta}_\Omega^{\text{op}}(y) \otimes 1) (\hat{W}_\Omega)_{21}^* = y \otimes 1 \otimes 1.$$

It follows that A_Ω is generated by elements of the form $(\omega \otimes \iota \otimes \iota) \eta_\Omega(\alpha(a))$ for $a \in A$ and $\omega \in K^*$. Since by (2.4)

$$\hat{W}_\Omega = (\tilde{J} \otimes \hat{J}) \Omega \hat{W}^* (J \otimes \hat{J}) \Omega^* = (\tilde{J} \otimes \hat{J}) \Omega (J \otimes \hat{J}) \hat{W} \Omega^*,$$

we can also write

$$\eta_\Omega(\alpha(a)) = \text{Ad}((\tilde{J} \otimes \hat{J}) \Omega (J \otimes \hat{J}))_{21} (\Delta \otimes \iota) \alpha(a).$$

It would be natural to replace \tilde{J} in the above expression by J , as it would not change A_Ω up to isomorphism, but we will keep the above formula to make some later identities look better.

As a first example consider $A = C_0(G)$ with the action of G on itself by left translations, so $\alpha = \Delta$.

Lemma 3.2. We have $C_0(G)_\Omega = \beta(C_r^*(\hat{G}; \Omega)) \cong C_r^*(\hat{G}; \Omega)$.

This lemma gives us a new set of generators of $C_r^*(\hat{G}; \Omega)$, which allows us to finally prove Proposition 2.1.

Proof of Lemma 3.2 and Proposition 2.1. For the proof we need the following identity:

$$(\hat{W}_\Omega \Omega)_{23} \hat{W}_{12} (\hat{W}_\Omega \Omega)_{23}^* = (\hat{W}_\Omega^*)_{12} (\hat{W}_\Omega \Omega)_{13}. \quad (3.1)$$

In order to prove this, we write identity (2.2) for the cocycle Ω^* on \hat{G}_Ω as

$$(\hat{W}_\Omega \Omega)_{23} (\hat{W}_\Omega \Omega)_{12} (\hat{W}_\Omega \Omega)_{23}^* = (\hat{W}_\Omega)_{12} (\hat{W}_\Omega \Omega)_{13}.$$

Substituting $(\hat{W}_\Omega)_{12}$ on both sides of the above identity for $((\tilde{J} \otimes \hat{J}) \Omega (J \otimes \hat{J}) \hat{W} \Omega^*)_{12}$ we get

$$(\hat{W}_\Omega \Omega)_{23} ((\tilde{J} \otimes \hat{J}) \Omega (J \otimes \hat{J}))_{12} \hat{W}_{12} (\hat{W}_\Omega \Omega)_{23}^* = ((\tilde{J} \otimes \hat{J}) \Omega (J \otimes \hat{J}))_{12} (\hat{W}_\Omega^*)_{12} (\hat{W}_\Omega \Omega)_{13}.$$

Since $(\hat{W}_\Omega \Omega)_{23}$ and $((\tilde{J} \otimes \hat{J}) \Omega (J \otimes \hat{J}))_{12}$ commute, this is exactly (3.1).

Using (3.1) we compute:

$$\begin{aligned} (\hat{W}_\Omega \Omega)_{32}(\iota \otimes \Delta)(\hat{W})_{134}(\hat{W}_\Omega \Omega)_{32}^* &= (\hat{W}_\Omega \Omega)_{32} \hat{W}_{14} \hat{W}_{13} (\hat{W}_\Omega \Omega)_{32}^* \\ &= \hat{W}_{14} (\hat{W}_\Omega^*)_{13} (\hat{W}_\Omega \Omega)_{12} \\ &= (\iota \otimes \beta)(\hat{W}_\Omega^*)_{134} (\hat{W}_\Omega \Omega)_{12}. \end{aligned}$$

Applying the slice maps to the first two legs we get

$$[(\omega \otimes \iota \otimes \iota) \eta_\Omega(\Delta(C_0(G))) \mid \omega \in K^*] = \beta([\omega \otimes \iota)(\hat{W}_\Omega^*) \mid \omega \in K^*].$$

This proves Lemma 3.2. Furthermore, the space on the left is clearly self-adjoint. Hence the same is true for $[(\omega \otimes \iota)(\hat{W}_\Omega^*) \mid \omega \in K^*]$. Since the latter space is an algebra, it coincides with $C_r^*(\hat{G}; \Omega)$, which proves Proposition 2.1. \square

The action β of G on $W^*(\hat{G}; \Omega)$ was defined in [32] as a dual action on a twisted crossed product. From the perspective of the previous lemma it can be viewed as the right action of G on itself that survives under deformation. More precisely, we have the following.

Proposition 3.3. *Assume A is a C*-algebra equipped with a continuous left action α of a locally compact quantum group G and a continuous right action γ of a locally compact quantum group H such that $(\iota \otimes \gamma)\alpha = (\alpha \otimes \iota)\gamma$. Then the restriction of $\iota \otimes \gamma: M(K \otimes A) \rightarrow M(K \otimes A \otimes C_0(H))$ to A_Ω defines a continuous right action of H on A_Ω .*

Proof. For $\omega \in K^*$ we have

$$\begin{aligned} [(1 \otimes 1 \otimes C_0(H))(\iota \otimes \gamma)((\omega \otimes \iota \otimes \iota) \eta_\Omega(\alpha(A)))] &= [(\omega \otimes \iota \otimes \iota \otimes \iota)(\eta_\Omega \otimes \iota)(\alpha \otimes \iota)((1 \otimes C_0(H))\gamma(A))] \\ &= [(\omega \otimes \iota \otimes \iota) \eta_\Omega(\alpha(A))] \otimes C_0(H). \end{aligned}$$

This implies that

$$[(1 \otimes 1 \otimes C_0(H))(\iota \otimes \gamma)(A_\Omega)] = A_\Omega \otimes C_0(H).$$

From this we conclude that $(\iota \otimes \gamma)(A_\Omega) \subset M(A_\Omega \otimes C_0(H))$ and the restriction of $\iota \otimes \gamma$ to A_Ω defines a continuous action of H . \square

Proof of Proposition 2.5. Applying the above proposition to $A = C_0(G)$ and $\gamma = \Delta$ we conclude that the restriction of $\iota \otimes \Delta$ to $C_0(G)_\Omega = \beta(C_r^*(\hat{G}; \Omega))$ defines a continuous right action of G . Since $(\iota \otimes \Delta)\beta = (\beta \otimes \iota)\beta$, this exactly means that $\beta|_{C_r^*(\hat{G}; \Omega)}$ defines a continuous action of G on $C_r^*(\hat{G}; \Omega)$. \square

By definition the Ω -deformation of a C*-algebra A lies in the multiplier algebra of $K \otimes A$. Under mild additional assumptions we have a more precise result.

Proposition 3.4. *Assume Ω is an admissible cocycle. Then*

$$A_\Omega \subset M(C_r^*(\hat{G}; \Omega) \otimes A) \quad \text{and} \quad (\beta \otimes \iota)(x) = (\iota \otimes \alpha)(x) \quad \text{for any } x \in A_\Omega.$$

Proof. By admissibility we have

$$(\hat{W}_\Omega^*)_{13}(\hat{W}_\Omega \Omega)_{12} \in M(K \otimes \hat{J}C_r^*(\hat{G}; \Omega)\hat{J} \otimes C_r^*(\hat{G}; \Omega)).$$

By virtue of identity (3.1) this is equivalent to

$$(\hat{W}_\Omega \Omega)_{32} \hat{W}_{13} (\hat{W}_\Omega \Omega)_{32}^* \in M(K \otimes \hat{J}C_r^*(\hat{G}; \Omega)\hat{J} \otimes C_r^*(\hat{G}; \Omega)).$$

Applying the slice maps to the first leg we conclude that $(\hat{W}_\Omega \Omega)_{21}(1 \otimes C_0(G))(\hat{W}_\Omega \Omega)_{21}^*$ is a non-degenerate C*-subalgebra of $M(\hat{J}C_r^*(\hat{G}; \Omega)\hat{J} \otimes C_r^*(\hat{G}; \Omega))$. It follows that $\eta_\Omega \alpha$ maps A into

$$M(\hat{J}C_r^*(\hat{G}; \Omega)\hat{J} \otimes C_r^*(\hat{G}; \Omega) \otimes A).$$

Applying the slice maps to the first leg we see that $A_\Omega \subset M(C_r^*(\hat{G}; \Omega) \otimes A)$.

If $a \in A$ and $\omega \in K^*$, then for $x = (\omega \otimes \iota \otimes \iota)\eta_\Omega(\alpha(a))$ we have

$$\begin{aligned} (\beta \otimes \iota)(x) &= V_{12}x_{13}V_{12}^* = (\omega \otimes \iota \otimes \iota \otimes \iota)(V_{23}(\hat{W}_\Omega\Omega)_{21}\alpha(a)_{24}(\hat{W}_\Omega\Omega)_{21}^*V_{23}^*) \\ &= (\omega \otimes \iota \otimes \iota \otimes \iota)((\hat{W}_\Omega\Omega)_{21}(\Delta \otimes \iota)(\alpha(a))_{234}(\hat{W}_\Omega\Omega)_{21}^*) = (\iota \otimes \alpha)(x), \end{aligned}$$

where we have used that $(\hat{W}_\Omega\Omega)_{21}$ commutes with V_{23} . \square

The above proposition shows that $C_r^*(\hat{G}; \Omega)$ plays a key role in the definition of Ω -deformation, as it should. To make this statement more precise, for any cocycle Ω , define the maps

$$T_\omega: C_0(G) \rightarrow C_r^*(\hat{G}; \Omega), \quad T_\omega(x) = (\omega \otimes \iota)((\hat{W}_\Omega\Omega)_{21}(1 \otimes x)(\hat{W}_\Omega\Omega)_{21}^*),$$

for $\omega \in W^*(\hat{G}_\Omega; \Omega^*)^*$. Note again that by (3.1) the image of T_ω is indeed contained in $C_r^*(\hat{G}; \Omega)$. Furthermore, by the proof of Lemma 3.2 the span of the images of these maps over all ω is dense in $C_r^*(\hat{G}; \Omega)$. These maps are right G -equivariant, with respect to the action Δ on $C_0(G)$ and the action β on $C_r^*(\hat{G}; \Omega)$. We then have

$$(\omega \otimes \iota \otimes \iota)\eta_\Omega(\alpha(a)) = (T_\omega \otimes \iota)\alpha(a) \quad \text{for } a \in A, \quad (3.2)$$

although this is a little bit ambiguous, since the expression on the right involves an extension of the completely bounded map $T_\omega \otimes \iota$ to $M(C_0(G) \otimes A)$. Therefore $A_\Omega \subset M(K \otimes A)$ is the C^* -algebra generated by $(T_\omega \otimes \iota)\alpha(A)$ for $\omega \in W^*(\hat{G}_\Omega; \Omega^*)^*$.

Note also that there exist maps going in the opposite direction, although we do not need them for the moment. Namely, for $\omega \in W^*(\hat{G}; \Omega)^*$ we can define

$$S_\omega: C_r^*(\hat{G}; \Omega) \rightarrow C_0(G), \quad S_\omega(x) = (\omega \otimes \iota)\beta(x).$$

Since the elements of the form $T_\nu(x)$, with $x \in C_0(G)$ and $\nu \in W^*(\hat{G}_\Omega; \Omega^*)^*$, span a dense subspace of $C_r^*(\hat{G}; \Omega)$, the following computation shows that the image of S_ω is contained in $C_0(G)$:

$$S_\omega T_\nu(x) = (\omega \otimes \iota)(T_\nu \otimes \iota)\Delta(x) = (\omega \otimes \nu \otimes \iota)((W_\Omega\Omega)_{21}(1 \otimes \Delta(x))(\hat{W}_\Omega\Omega)_{21}^*) \in C_0(G).$$

The maps S_ω are again right G -equivariant.

In the context of the Rieffel deformation the maps T_ν and S_ω , with ν and ω taken to be the vacuum states, played an important role in [3].

3.2. Relation to twisted crossed products. Assume B is a C^* -algebra and γ is a continuous left action of \hat{G}^{op} on B . Consider the crossed product

$$\hat{G}^{\text{op}} \ltimes_\gamma B = [(JC_0(G)J \otimes 1)\gamma(B)]$$

and the dual action $\alpha = \hat{\gamma}$ of G on $\hat{G}^{\text{op}} \ltimes_\gamma B$ given by

$$\alpha(x) = \text{Ad}((1 \otimes J\hat{J} \otimes 1)(W^* \otimes 1)(1 \otimes \hat{J}J \otimes 1))(1 \otimes x).$$

Generalizing Lemma 3.2 we then have the following.

Proposition 3.5. *We have $(\hat{G}^{\text{op}} \ltimes_\gamma B)_\Omega \cong \hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B$. More precisely,*

$$\text{Ad}((1 \otimes J\hat{J} \otimes 1)(\hat{W}_\Omega\Omega^*)_{21}^*(1 \otimes \hat{J}J \otimes 1))$$

maps the Ω -deformation of $\hat{G}^{\text{op}} \ltimes_\gamma B$ onto $1 \otimes (\hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B)$.

Proof. The conjugation by $\hat{J}J$ defines a left G -equivariant isomorphism $JC_0(G)J \cong C_0(G)$. Therefore by Lemma 3.2 the Ω -deformation of $JC_0(G)J$ is equal to

$$(1 \otimes J\hat{J})\beta(C_r^*(\hat{G}; \Omega))(1 \otimes \hat{J}J).$$

From this, by definition of the Ω -deformation, we conclude that $(\hat{G}^{\text{op}} \ltimes_\gamma B)_\Omega$ is generated by

$$(1 \otimes J\hat{J} \otimes 1)\beta(C_r^*(\hat{G}; \Omega)) \otimes 1)(1 \otimes \hat{J}J \otimes 1)(1 \otimes \gamma(B)).$$

Now recall that $\beta(x) = (\hat{W}\Omega^*)_{21}(1 \otimes x)(\hat{W}\Omega^*)_{21}^*$ by (2.3). Observing also that the unitary

$$(1 \otimes J\hat{J} \otimes 1)(\hat{W}\Omega^*)_{21}^*(1 \otimes \hat{J}J \otimes 1)$$

commutes with $1 \otimes \gamma(B)$, we conclude that the conjugation by this unitary maps $(\hat{G}^{\text{op}} \ltimes_{\gamma} B)_{\Omega}$ onto the C*-algebra generated by $(1 \otimes J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1)(1 \otimes \gamma(B))$. But this is exactly $1 \otimes (\hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B)$. \square

Turning to more general actions, recall that for regular quantum groups any action is stably exterior equivalent to a dual action. Therefore it is natural to expect that up to stabilization Ω -deformations can be expressed in terms of twisted crossed products, at least under some regularity assumptions. In order to formulate the result recall that in Sections 2.3 and 2.4 we already used the unitaries

$$X = \tilde{J}J \in L^{\infty}(\hat{G}) \quad \text{and} \quad Y = \hat{V}_{21}^* \Omega_{21}^* = (J \otimes \tilde{J})\hat{W}\Omega^*(\tilde{J} \otimes \tilde{J}).$$

Using formula (2.4) for \hat{W}_{Ω} we can also write

$$Y = (1 \otimes \tilde{J}\hat{J})(\hat{W}_{\Omega}\Omega)^*(1 \otimes \hat{J}\tilde{J}).$$

Theorem 3.6. *Assume Ω is a regular cocycle on a locally compact quantum group \hat{G} . Then for any C*-algebra A equipped with a continuous left action α of G we have*

$$\hat{G}^{\text{op}} \ltimes_{\hat{\alpha}, \Omega} G \ltimes_{\alpha} A \cong K \otimes A_{\Omega}.$$

Explicitly, the map

$$\text{Ad}((\hat{J}J \otimes 1 \otimes 1)Y_{21}^*(J\hat{J} \otimes 1 \otimes 1)) = \text{Ad}((\hat{J}X^*\hat{J} \otimes 1 \otimes 1)(\hat{W}_{\Omega}\Omega)_{21}(\hat{J}X\hat{J} \otimes 1 \otimes 1))$$

defines such an isomorphism. This map is trivial on $J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1 \otimes 1$ and it maps $\hat{\Delta}^{\text{op}}(x) \otimes 1 \in \hat{\Delta}^{\text{op}}(C_0(\hat{G})) \otimes 1$ into $x \otimes 1 \otimes 1$ and $1 \otimes \alpha(a) \in 1 \otimes \alpha(A)$ into $\text{Ad}(\hat{J}X^*\hat{J} \otimes 1 \otimes 1)\eta_{\Omega}(\alpha(a))$.

We will simultaneously prove the following.

Theorem 3.7. *If Ω is regular, then*

$$A_{\Omega} = [(\omega \otimes \iota \otimes \iota)\eta_{\Omega}(\alpha(A)) \mid \omega \in K^*].$$

Proof of Theorems 3.6 and 3.7. From the equality $Y = (J \otimes \tilde{J})\hat{W}\Omega^*(\tilde{J} \otimes \tilde{J})$ it is clear that Y commutes with $1 \otimes C_r^*(\hat{G}; \Omega)$. Hence the homomorphism in the formulation is trivial on $J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}J \otimes 1 \otimes 1$. Clearly, it maps $1 \otimes \alpha(a) \in 1 \otimes \alpha(A)$ into $\text{Ad}(\hat{J}X^*\hat{J} \otimes 1 \otimes 1)\eta_{\Omega}(\alpha(a))$. Finally, for $x \in C_0(\hat{G})$ this homomorphism maps $\hat{\Delta}^{\text{op}}(x) \otimes 1$ into

$$(\hat{W}_{\Omega}\Omega)_{21}(\hat{\Delta}^{\text{op}}(x) \otimes 1)(\hat{W}_{\Omega}\Omega^*)_{21} = (\hat{W}_{\Omega}\Omega\hat{\Delta}(x)(\hat{W}_{\Omega}\Omega)^*)_{21} = (\hat{W}_{\Omega}\hat{\Delta}_{\Omega}(x)\hat{W}_{\Omega}^*)_{21} = x \otimes 1 \otimes 1.$$

Since $[J\hat{J}C_r^*(\hat{G}; \Omega)\hat{J}JC_0(\hat{G})] = K$ by regularity of Ω , it follows that this homomorphism maps the C*-algebra $\hat{G}^{\text{op}} \ltimes_{\hat{\alpha}, \Omega} G \ltimes_{\alpha} A$ onto

$$[(K \otimes 1 \otimes 1)\text{Ad}(\hat{J}X^*\hat{J} \otimes 1 \otimes 1)\eta_{\Omega}(\alpha(A))].$$

In particular, the last space is a C*-algebra, so it coincides with

$$[(K \otimes 1 \otimes 1)\text{Ad}(\hat{J}X^*\hat{J} \otimes 1 \otimes 1)\eta_{\Omega}(\alpha(A))(K \otimes 1 \otimes 1)] = K \otimes [(\omega \otimes \iota \otimes \iota)\eta_{\Omega}(\alpha(A)) \mid \omega \in K^*].$$

This shows that $[(\omega \otimes \iota \otimes \iota)\eta_{\Omega}(\alpha(A)) \mid \omega \in K^*]$ is a C*-algebra and finishes the proof of both theorems. \square

Note that while the regularity of Ω is a necessary condition for the conclusion of Theorem 3.6 to be true, it is not clear whether this is the case for Theorem 3.7. For example, by the proof of Proposition 3.5, for dual actions Theorem 3.7 remains true for any Ω .

3.3. Deformed action. Our motivation for defining the Ω -deformation was that in good situations the deformed dual action $\hat{\alpha}_\Omega$ should be dual to some action α_Ω . It turns out that α_Ω is well-defined under more general assumptions than $\hat{\alpha}_\Omega$.

Theorem 3.8. *Assume Ω is a measurable unitary 2-cocycle on a locally compact quantum group \hat{G} such that the deformed quantum group G_Ω is regular. Then for any C^* -algebra A equipped with a continuous left action α of G , the formula $\alpha_\Omega(x) = W_\Omega^*(1 \otimes x)W_\Omega$ defines a continuous left action of G_Ω on A_Ω .*

Proof. This follows from the proof of [31, Theorem 6.7]. We include a complete argument for the reader's convenience. Using the identity $(V_\Omega)_{23}(\hat{W}_\Omega)_{12}(V_\Omega)_{23}^* = (\hat{W}_\Omega)_{13}(\hat{W}_\Omega)_{12}^*$, for $a \in A$ we compute:

$$\begin{aligned} (\iota \otimes \alpha_\Omega)\eta_\Omega(\alpha(a)) &= (\hat{W}_\Omega)_{32}(\hat{W}_\Omega\Omega)_{31}\alpha(a)_{34}(\hat{W}_\Omega\Omega)_{31}^*(\hat{W}_\Omega)_{32}^* \\ &= (V_\Omega)_{12}(\hat{W}_\Omega)_{31}(V_\Omega)_{12}^*\Omega_{31}\alpha(a)_{34}\Omega_{31}^*(V_\Omega)_{12}(\hat{W}_\Omega)_{31}^*(V_\Omega)_{12}^*. \end{aligned}$$

Since $(V_\Omega)_{12}^*$ and Ω_{31} commute, we thus get

$$(\iota \otimes \alpha_\Omega)\eta_\Omega(\alpha(a)) = (V_\Omega)_{12}\eta_\Omega(\alpha(a))_{134}(V_\Omega)_{12}^*.$$

Multiplying this identity on the left by $1 \otimes C_0(G_\Omega) \otimes 1 \otimes 1$, applying the slice maps to the first leg and using that $V_\Omega \in M(K \otimes C_0(G_\Omega))$ and $[(1 \otimes C_0(G_\Omega))V_\Omega^*(K \otimes 1)] = K \otimes C_0(G_\Omega)$ by regularity, we see that

$$[(C_0(G_\Omega) \otimes 1 \otimes 1)\alpha_\Omega([\omega \otimes \iota \otimes \iota)\eta_\Omega(\alpha(A)) \mid \omega \in K^*]) = C_0(G_\Omega) \otimes [(\omega \otimes \iota \otimes \iota)\eta_\Omega(\alpha(A)) \mid \omega \in K^*].$$

This implies that $\alpha_\Omega(A_\Omega) \subset M(C_0(G_\Omega) \otimes A_\Omega)$ and that the cancelation property holds. Finally, it is easy to check that $(\iota \otimes \alpha_\Omega)\alpha_\Omega = (\Delta_\Omega \otimes \iota)\alpha_\Omega$. \square

Under more assumptions we can apply [31, Theorem 6.7] and get the following.

Theorem 3.9. *Under the assumptions of the previous theorem suppose also that the formula*

$$\hat{\alpha}_\Omega(x) = \Omega_{21}\hat{\alpha}(x)\Omega_{21}^*$$

defines a continuous action of $(\hat{G}_\Omega)^{\text{op}}$ on $G \rtimes_\alpha A$ and that $W_\Omega \otimes 1 \in M(C_0(G_\Omega) \otimes (G \rtimes_\alpha A))$. Then $G \rtimes_\alpha A = [(C_0(\hat{G}_\Omega) \otimes 1)A_\Omega]$ and the map η_Ω defines an isomorphism $G \rtimes_\alpha A \cong G_\Omega \rtimes_{\alpha_\Omega} A_\Omega$. Under this isomorphism the deformed dual action $\hat{\alpha}_\Omega$ on $G \rtimes_\alpha A$ becomes the action dual to α_Ω .

Again, it is not clear to us what the optimal assumptions for the above two theorems are. Note, however, that for non-regular quantum groups it is not even obvious what the correct definition of a continuous action should be, see the discussion in [2]. Even if G_Ω is regular, it is doubtful that the map η_Ω defines an isomorphism $G \rtimes_\alpha A \cong G_\Omega \rtimes_{\alpha_\Omega} A_\Omega$ for any A , since this would imply that the deformed dual action $\hat{\alpha}_\Omega$ is well-defined on $G \rtimes_\alpha A$, which seems to be overly optimistic already when \hat{G} is a group and Ω is a measurable, but not continuous, cocycle on \hat{G} .

3.4. Deformation in stages. If Ω is a cocycle on \hat{G} and Ω_1 is a cocycle on \hat{G}_Ω , then it is easy to check that $\Omega_1\Omega$ is a cocycle on \hat{G} . Therefore if the deformed action α_Ω of G_Ω on A_Ω is well-defined, then we can compare the Ω_1 -deformation of A_Ω with the $\Omega_1\Omega$ -deformation of A .

Theorem 3.10. *Assume G is a locally compact quantum group, Ω is a measurable unitary 2-cocycle on \hat{G} , Ω_1 is a measurable unitary 2-cocycle on \hat{G}_Ω , and A is a C^* -algebra equipped with a continuous left action α of G . Suppose the following conditions are satisfied:*

- (i) $A_\Omega = [(\omega \otimes \iota \otimes \iota)\eta_\Omega(\alpha(A)) \mid \omega \in K^*]$;
- (ii) *the deformed action α_Ω of G_Ω is well-defined on A_Ω .*

Then the map $x \mapsto (\hat{W}_\Omega\Omega_1^)_{21}(1 \otimes x)(\hat{W}_\Omega\Omega_1^*)_{21}^*$ defines an isomorphism $A_{\Omega_1\Omega} \cong (A_\Omega)_{\Omega_1}$. Furthermore, if one of the deformed actions $\alpha_{\Omega_1\Omega}$ and $(\alpha_\Omega)_{\Omega_1}$ is well-defined, then the other is also well-defined and the isomorphism $A_{\Omega_1\Omega} \cong (A_\Omega)_{\Omega_1}$ is $G_{\Omega_1\Omega}$ -equivariant.*

Proof. For the proof we need the following identity:

$$(\hat{W}_\Omega \Omega)_{23}(\hat{W}_{\Omega_1 \Omega} \Omega_1 \Omega)_{12}(\hat{W}_\Omega \Omega)_{23}^* = (\hat{W}_{\Omega_1 \Omega} \Omega_1)_{12}(\hat{W}_\Omega \Omega)_{13}. \quad (3.3)$$

In order to show this, similarly to the proof of (3.1) we start with the identity

$$(\hat{W}_\Omega \Omega)_{23}(\hat{W}_\Omega \Omega)_{12}(\hat{W}_\Omega \Omega)_{23}^* = (\hat{W}_\Omega)_{12}(\hat{W}_\Omega \Omega)_{13}.$$

Next, substitute $(\hat{W}_\Omega)_{12}$ on both sides for its expression involving $(\hat{W}_{\Omega_1 \Omega})_{12}$ obtained from (2.4):

$$\hat{W}_{\Omega_1 \Omega} = (\tilde{J}_1 \otimes \hat{J})\Omega_1(J_\Omega \otimes \hat{J})\hat{W}_\Omega \Omega_1^*,$$

where \tilde{J}_1 is the modular involution defined by the dual weight on $W^*(\hat{G}_\Omega; \Omega_1)$ as explained in Section 2.2. We get

$$\begin{aligned} (\hat{W}_\Omega \Omega)_{23}((\tilde{J}_1 \otimes \hat{J})\Omega_1(J_\Omega \otimes \hat{J}))_{12}^*(\hat{W}_{\Omega_1 \Omega} \Omega_1 \Omega)_{12}(\hat{W}_\Omega \Omega)_{23}^* \\ = ((\tilde{J}_1 \otimes \hat{J})\Omega_1(J_\Omega \otimes \hat{J}))_{12}^*((\hat{W}_{\Omega_1 \Omega} \Omega_1)_{12}(\hat{W}_\Omega \Omega)_{13}). \end{aligned}$$

Since $(\hat{W}_\Omega \Omega)_{23}$ and $((\tilde{J}_1 \otimes \hat{J})\Omega_1(J_\Omega \otimes \hat{J}))_{12}^*$ commute, this is exactly (3.3).

For $a \in A$ we now start computing:

$$(\iota \otimes \eta_{\Omega_1} \alpha_\Omega) \eta_\Omega(\alpha(a)) = (\hat{W}_{\Omega_1 \Omega} \Omega_1)_{32}(\hat{W}_\Omega)_{43} \eta_\Omega(\alpha(a))_{145} (\hat{W}_\Omega)_{43}^* (\hat{W}_{\Omega_1 \Omega} \Omega_1)_{32}^*.$$

By identity (3.1), applied to the quantum group G_Ω and the dual cocycle Ω_1 , we have

$$(\hat{W}_{\Omega_1 \Omega} \Omega_1)_{32}(\hat{W}_\Omega)_{43}(\hat{W}_{\Omega_1 \Omega} \Omega_1)_{32}^* = (\hat{W}_\Omega \Omega_1^*)_{43}(\hat{W}_{\Omega_1 \Omega} \Omega_1)_{42}.$$

Therefore

$$\begin{aligned} (\iota \otimes \eta_{\Omega_1} \alpha_\Omega) \eta_\Omega(\alpha(a)) &= (\hat{W}_\Omega \Omega_1^*)_{43}(\hat{W}_{\Omega_1 \Omega} \Omega_1)_{42} \eta_\Omega(\alpha(a))_{145} (\hat{W}_{\Omega_1 \Omega} \Omega_1)_{42}^* (\hat{W}_\Omega \Omega_1^*)_{43}^* \\ &= (\hat{W}_\Omega \Omega_1^*)_{43}(\hat{W}_{\Omega_1 \Omega} \Omega_1)_{42}(\hat{W}_\Omega \Omega)_{41} \alpha(a)_{45} (\hat{W}_\Omega \Omega)_{41}^* (\hat{W}_{\Omega_1 \Omega} \Omega_1)_{42}^* (\hat{W}_\Omega \Omega_1^*)_{43}^*. \end{aligned}$$

By (3.3) the last expression equals

$$\begin{aligned} (\hat{W}_\Omega \Omega_1^*)_{43}(\hat{W}_\Omega \Omega)_{21}(\hat{W}_{\Omega_1 \Omega} \Omega_1 \Omega)_{42}(\hat{W}_\Omega \Omega)_{21}^* \alpha(a)_{45} (\hat{W}_\Omega \Omega)_{21}(\hat{W}_{\Omega_1 \Omega} \Omega_1 \Omega)_{42}^* (\hat{W}_\Omega \Omega)_{21}^* (\hat{W}_\Omega \Omega_1^*)_{43}^* \\ = (\hat{W}_\Omega \Omega_1^*)_{43}(\hat{W}_\Omega \Omega)_{21}(\hat{W}_{\Omega_1 \Omega} \Omega_1 \Omega)_{42} \alpha(a)_{45} (\hat{W}_{\Omega_1 \Omega} \Omega_1 \Omega)_{42}^* (\hat{W}_\Omega \Omega)_{21}^* (\hat{W}_\Omega \Omega_1^*)_{43}^* \\ = (\hat{W}_\Omega \Omega_1^*)_{43}(\hat{W}_\Omega \Omega)_{21} \eta_{\Omega_1 \Omega}(\alpha(a))_{245} (\hat{W}_\Omega \Omega)_{21}^* (\hat{W}_\Omega \Omega_1^*)_{43}^*. \end{aligned}$$

Thus

$$(\iota \otimes \eta_{\Omega_1} \alpha_\Omega) \eta_\Omega(\alpha(a)) = (\hat{W}_\Omega \Omega)_{21}(\hat{W}_\Omega \Omega_1^*)_{43} \eta_{\Omega_1 \Omega}(\alpha(a))_{245} (\hat{W}_\Omega \Omega_1^*)_{43}^* (\hat{W}_\Omega \Omega)_{21}^*.$$

Applying the slice maps to the first two legs we get the first statement of the theorem.

In order to show that the isomorphism $A_{\Omega_1 \Omega} \cong (A_\Omega)_{\Omega_1}$ is $G_{\Omega_1 \Omega}$ -equivariant we need the identity

$$(\hat{W}_{\Omega_1 \Omega})_{23}(\hat{W}_\Omega \Omega_1^*)_{12} = (\hat{W}_\Omega \Omega_1^*)_{12}(\hat{W}_{\Omega_1 \Omega})_{13}(\hat{W}_{\Omega_1 \Omega})_{23}.$$

Since $\hat{W}_\Omega \Omega_1^* = ((\tilde{J}_1 \otimes \hat{J})\Omega_1(J_\Omega \otimes \hat{J}))^* \hat{W}_{\Omega_1 \Omega}$, this is simply the pentagon relation for $\hat{W}_{\Omega_1 \Omega}$. Denoting the isomorphism in the formulation of the theorem by θ we compute:

$$\begin{aligned} (\alpha_\Omega)_{\Omega_1}(\theta(x)) &= (\hat{W}_{\Omega_1 \Omega})_{21}(1 \otimes \theta(x))(\hat{W}_{\Omega_1 \Omega})_{21}^* \\ &= (\hat{W}_{\Omega_1 \Omega})_{21}(\hat{W}_\Omega \Omega_1^*)_{32}(1 \otimes 1 \otimes x)(\hat{W}_\Omega \Omega_1^*)_{32}^* (\hat{W}_{\Omega_1 \Omega})_{21}^* \\ &= (\hat{W}_\Omega \Omega_1^*)_{32}(\hat{W}_{\Omega_1 \Omega})_{31}(\hat{W}_{\Omega_1 \Omega})_{21}(1 \otimes 1 \otimes x)(\hat{W}_{\Omega_1 \Omega})_{21}^* (\hat{W}_{\Omega_1 \Omega})_{31}^* (\hat{W}_\Omega \Omega_1^*)_{32}^* \\ &= (\hat{W}_\Omega \Omega_1^*)_{32} \alpha_{\Omega_1 \Omega}(x)_{13} (\hat{W}_\Omega \Omega_1^*)_{32}^* = (\iota \otimes \theta) \alpha_{\Omega_1 \Omega}(x). \end{aligned}$$

This proves the second statement of the theorem. \square

Example 3.11.

(i) It is straightforward to check that the deformation A_1 of A with respect to the trivial cocycle 1 is $\alpha(A)$. Therefore given a cocycle Ω such that the assumptions (i) and (ii) in the above theorem are satisfied, it follows that the map $x \mapsto (\hat{W}_\Omega \Omega)_{21}(1 \otimes x)(\hat{W}_\Omega \Omega)_{21}^*$ defines an isomorphism $\alpha(A) \cong (A_\Omega)_\Omega^*$. In other words, the map $\eta_\Omega \alpha$ is an isomorphism $A \cong (A_\Omega)_\Omega^*$.

(ii) Assume $A = \hat{G}^{\text{op}} \ltimes_\gamma B$ and $\alpha = \hat{\gamma}$. Then by Proposition 3.5 we have $A_\Omega \cong \hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B$. By Proposition 2.9 we have a dual action on $\hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B$. It is easy to check that this is exactly the deformed action α_Ω . Furthermore, as we have already remarked, the proof of Proposition 3.5 shows that $A_\Omega = [(\omega \otimes \iota \otimes \iota)\eta_\Omega(\alpha(A)) \mid \omega \in K^*]$. Therefore for dual actions conditions (i) and (ii) in the above theorem are always satisfied. For any cocycle Ω_1 on \hat{G}_Ω we thus get

$$(\hat{G}^{\text{op}} \ltimes_{\gamma, \Omega} B)_{\Omega_1} \cong (\hat{G}^{\text{op}} \ltimes_\gamma B)_{\Omega_1 \Omega} \cong \hat{G}^{\text{op}} \ltimes_{\gamma, \Omega_1 \Omega} B.$$

In particular, for the C^* -algebra $C_r^*(\hat{G}; \Omega)$ equipped with the action β_Ω of G_Ω we get $C_r^*(\hat{G}; \Omega)_{\Omega_1} \cong C_r^*(\hat{G}; \Omega_1 \Omega)$. \diamond

We finish our general discussion of Ω -deformations with the observation that up to isomorphism the C^* -algebra A_Ω depends only on the cohomology class of Ω . Recall that given a 2-cocycle Ω on \hat{G} and a unitary $u \in L^\infty(\hat{G})$, the element $\Omega_u = (u \otimes u)\Omega\hat{\Delta}(u)^*$ is again a cocycle on \hat{G} . The cocycles Ω and Ω_u are called cohomologous. The set of cohomology classes of unitary 2-cocycles on \hat{G} is denoted by $H^2(\hat{G}; \mathbb{T})$. In general it is just a set.

Proposition 3.12. *Assume G is a locally compact quantum group, Ω is a measurable unitary 2-cocycle on \hat{G} and u is a unitary in $L^\infty(\hat{G})$. Then for any C^* -algebra A equipped with a continuous left action of G , the map $\text{Ad}(u \otimes 1)$ defines an isomorphism $A_\Omega \cong A_{\Omega_u}$.*

Proof. Since $\hat{W}\Omega_u^* = \hat{W}\hat{\Delta}(u)\Omega^*(u^* \otimes u^*) = (1 \otimes u)\hat{W}\Omega^*(u^* \otimes u^*)$, the map $\text{Ad } u$ defines a right G -equivariant isomorphism $W^*(\hat{G}; \Omega) \cong W^*(\hat{G}; \Omega_u)$. This isomorphism maps $(\omega(\cdot u^*) \otimes \iota)(\hat{W}\Omega^*)$ into $(\omega \otimes \iota)(\hat{W}\Omega_u^*)$. Consider the GNS-representations $\tilde{\Lambda}: \mathcal{N}_{\tilde{\varphi}} \rightarrow L^2(G)$ and $\tilde{\Lambda}_u: \mathcal{N}_{\tilde{\varphi}_u} \rightarrow L^2(G)$ defined by the dual weights $\tilde{\varphi}$ on $W^*(\hat{G}; \Omega)$ and $\tilde{\varphi}_u$ on $W^*(\hat{G}; \Omega_u)$, as described in Section 2.2. Then the isomorphism $W^*(\hat{G}; \Omega) \cong W^*(\hat{G}; \Omega_u)$ defines a unitary \tilde{u} on $L^2(G)$ such that $\tilde{u}\tilde{\Lambda}(x) = \tilde{\Lambda}_u(xu^*)$ for $x \in \mathcal{N}_{\tilde{\varphi}}$, so

$$\tilde{u}\tilde{\Lambda}((\omega(\cdot u^*) \otimes \iota)(\hat{W}\Omega^*)) = \tilde{\Lambda}_u((\omega \otimes \iota)(\hat{W}\Omega_u^*))$$

for suitable $\omega \in K^*$. Since $\tilde{\Lambda}((\omega(\cdot u^*) \otimes \iota)(\hat{W}\Omega^*)) = \Lambda((\omega(\cdot u^*) \otimes \iota)(\hat{W}))$ and a similar formula holds for $\tilde{\Lambda}_u$, we in other words have

$$\tilde{u}\Lambda((\omega(\cdot u^*) \otimes \iota)(\hat{W})) = \Lambda(\omega \otimes \iota)(\hat{W}).$$

But $\Lambda((\omega(\cdot u^*) \otimes \iota)(\hat{W})) = u^*\Lambda((\omega \otimes \iota)(\hat{W}))$, as can be easily checked using that

$$(\Lambda((\nu \otimes \iota)(\hat{W})), \hat{\Lambda}(y)) = \nu(y^*).$$

It follows that $\tilde{u} = u$. Hence the modular involution \tilde{J}_u defined by the weight $\tilde{\varphi}_u$ on $W^*(\hat{G}; \Omega_u)$ is equal to $u\tilde{J}u^*$. Therefore

$$(\tilde{J}_u \otimes \hat{J})\Omega_u(J \otimes \hat{J}) = (u \otimes \hat{J}u\hat{J})((\tilde{J} \otimes \hat{J})\Omega(J \otimes \hat{J}))(J \otimes \hat{J})\hat{\Delta}(u)^*(J \otimes \hat{J}).$$

Next, we have

$$(J \otimes \hat{J})\hat{\Delta}(u)^*(J \otimes \hat{J})\hat{W} = (J \otimes \hat{J})\hat{\Delta}(u)^*\hat{W}^*(J \otimes \hat{J}) = (J \otimes \hat{J})\hat{W}^*(1 \otimes u^*)(J \otimes \hat{J}) = \hat{W}(1 \otimes \hat{J}u^*\hat{J}).$$

Hence, by (2.4),

$$\hat{W}_{\Omega_u}\Omega_u = (\tilde{J}_u \otimes \hat{J})\Omega_u(J \otimes \hat{J})\hat{W} = (u \otimes \hat{J}u\hat{J})\hat{W}_\Omega\Omega(1 \otimes \hat{J}u^*\hat{J}).$$

Recalling the definition of η_Ω we get

$$\eta_{\Omega_u} = \text{Ad}(\hat{J}u\hat{J} \otimes u \otimes 1)\eta_\Omega.$$

This gives the result. \square

4. COCYCLES ON GROUP DUALS

In this section we assume that G is a genuine locally compact group.

4.1. Dual cocycles and deformations of the Fourier algebra. Denote by λ_g , resp. ρ_g , the operators of the left, resp. right, regular representation of G . Then

$$(\hat{W}\xi)(s, t) = \xi(ts, t) = (\lambda_t^{-1}\xi(\cdot, t))(s) \quad \text{and} \quad (V\xi)(s, t) = (\rho_t\xi(\cdot, t))(s) \quad \text{for } \xi \in L^2(G \times G).$$

The predual of $L^\infty(\hat{G}) = W^*(G)$ can be identified with the Fourier algebra $A(G) \subset C_0(G)$, so an element $\omega \in W^*(G)_*$ is identified with the function $f(g) = \omega(\lambda_g)$ on G . Note that under this identification we have

$$(f \otimes \iota)(\hat{W}) = \check{f} \quad \text{for } f \in A(G),$$

where $\check{f}(g) = f(g^{-1})$.

Assume now that Ω is a measurable unitary 2-cocycle on \hat{G} . Then we can define a new product \star_Ω on $A(G)$ by

$$f_1 \star_\Omega f_2 = (f_1 \otimes f_2)(\hat{\Delta}(\cdot)\Omega^*).$$

The associativity of this product is equivalent to the cocycle identity for Ω . Identity (2.1) shows that the formula

$$\pi_\Omega(f) = (f \otimes \iota)(\hat{W}\Omega^*)$$

defines a representation of $(A(G), \star_\Omega)$ on $L^2(G)$, and then by definition the C*-algebra $C_r^*(\hat{G}; \Omega)$ is generated by $\pi_\Omega(A(G))$. Recall that by Proposition 2.1 we in fact have $C_r^*(\hat{G}; \Omega) = \overline{\pi_\Omega(A(G))}$. Nevertheless we do not claim that $(A(G), \star_\Omega)$ is itself a *-algebra, although this is often the case. In Section 5 we will give an example where $(A(G), \star_\Omega)$ does not inherit a *-structure from $\pi_\Omega(A(G)) \subset C_r^*(\hat{G}; \Omega)$. Since

$$\tilde{\Lambda}((f \otimes \iota)(\hat{W}\Omega^*)) = \Lambda((f \otimes \iota)(\hat{W})) = \check{f},$$

the representation π_Ω is given by

$$\pi_\Omega(f_1)\check{f}_2 = (f_1 \star_\Omega f_2)^\sim \quad \text{for } f_1 \in A(G) \quad \text{and} \quad f_2 \in A(G) \cap C_c(G).$$

In other words, since $\int_G \check{f}(g)dg = \int_G f(g)\Delta_G(g)^{-1}dg$, the representation π_Ω is simply the left regular representation of $(A(G), \star_\Omega)$ on itself, with $A(G)$, or more precisely $A(G) \cap C_c(G)$, completed to a Hilbert space using the scalar product defined by the right Haar measure $\Delta_G(g)^{-1}dg$.

The left translations of G on itself define automorphisms of $(A(G), \star_\Omega)$. On the level of $C_r^*(\hat{G}; \Omega)$ the action by left translations is exactly the action β introduced earlier, so

$$\beta_g(\pi_\Omega(f)) = \pi_\Omega(f(g^{-1}\cdot)) \quad \text{for } f \in A(G).$$

This is the reason for the appearance of the right Haar measure, since the average of a function on G with respect to the action by left translations is the integral with respect to a right Haar measure.

Conversely, assume we have a product \star on $A(G)$ that is invariant under left translations. Assume there exists an element $\Omega \in W^*(G) \hat{\otimes} W^*(G)$ such that

$$(f_1 \otimes f_2)(\Omega^*) = (f_1 \star f_2)(e) \quad \text{for all } f_1, f_2 \in A(G),$$

which happens exactly when the map $f_1 \otimes f_2 \mapsto (f_1 \star f_2)(e)$ extends to a bounded linear functional on the projective tensor product $A(G) \hat{\otimes} A(G)$. Then $f_1 \star f_2 = (f_1 \otimes f_2)(\hat{\Delta}(\cdot)\Omega^*)$. For finite groups this was observed by Movshev [23]. As follows from results in [23], if G is finite and $(A(G), \star)$ is semisimple, then Ω is invertible and cohomological to a unitary cocycle, that is, there exists an invertible element $a \in W^*(G)$ such that $(a \otimes a)\Omega\hat{\Delta}(a)^{-1}$ is unitary.

For a related discussion see [20].

4.2. Ω -Deformations and generalized fixed point algebras. As we know, by [32, Lemma 1.12] the action β of G on $W^*(\hat{G}; \Omega)$ is integrable. We will now show a stronger property: the action of G on $C_r^*(\hat{G}; \Omega)$ is integrable. Since the integrability property that we will establish appears under several different names in the literature, let us say precisely what we mean by this.

Assume γ is a continuous action of G on a C^* -algebra B . An element $b \in B_+$ is called γ -integrable if there exists an element $\psi_\gamma(b) \in M(B)$ such that for every state ω on B the function $g \mapsto \omega(\gamma_g(b))$ is integrable and

$$\int_G \omega(\gamma_g(b)) dg = \omega(\psi_\gamma(b)).$$

Clearly, the element $\psi_\gamma(b)$ is uniquely determined by b , and $\psi_\gamma(b) \in M(B)^\gamma$.

There are several other equivalent definitions of integrable elements, see [28] (note that in [28] γ -integrable elements are called γ -proper). For example, by [28, Proposition 4.4] a positive element b is γ -integrable if and only if the functions $g \mapsto \gamma_g(b)c$ and $g \mapsto c\gamma_g(b)$ are unconditionally integrable for all $c \in B$, meaning that their integrals over compact subsets of G form Cauchy nets.

The set \mathcal{P}_γ^+ of γ -integrable positive elements is a hereditary cone in B_+ , see e.g. [28, Lemma 2.7]. Hence the linear span of \mathcal{P}_γ^+ is a $*$ -algebra. We say that γ is integrable if \mathcal{P}_γ^+ is dense in B_+ , or equivalently, $[\mathcal{P}_\gamma^+] = B$.

Proposition 4.1. *For any measurable unitary 2-cocycle Ω on \hat{G} the action β of G on $C_r^*(\hat{G}; \Omega)$ is integrable.*

Before we turn to the proof, let us discuss the difference between this statement and the integrability of the action of G on $W^*(\hat{G}; \Omega)$. By [32, Theorem 1.11] the action on $W^*(\hat{G}; \Omega)$ is ergodic. In particular, $M(C_r^*(\hat{G}; \Omega))^\beta = \mathbb{C}1$. For $x \in W^*(\hat{G}; \Omega)_+$ in the domain of definition $\text{dom } \tilde{\varphi}$ of the dual weight $\tilde{\varphi}$ we have

$$\int_G \omega(\beta_g(x)) dg = \tilde{\varphi}(x) \quad (4.1)$$

for all normal states ω on $W^*(\hat{G}; \Omega)$. It follows that if $x \in C_r^*(\hat{G}; \Omega)_+$ is β -integrable, then $x \in \text{dom } \tilde{\varphi}$ and $\psi_\beta(x) = \tilde{\varphi}(x)1$. The difference between the cones \mathcal{P}_β^+ and $C_r^*(\hat{G}; \Omega) \cap \text{dom } \tilde{\varphi}$ is that for the elements of \mathcal{P}_β^+ identity (4.1) should be satisfied for all states ω on $C_r^*(\hat{G}; \Omega)$, while for the elements of $C_r^*(\hat{G}; \Omega) \cap \text{dom } \tilde{\varphi}$ we need only to consider normal states on $W^*(\hat{G}; \Omega)$. Note also that the density of $C_r^*(\hat{G}; \Omega) \cap \text{dom } \tilde{\varphi}$ in $C_r^*(\hat{G}; \Omega)_+$ follows already from the proof of [32, Lemma 1.12].

Proof of Proposition 4.1. For a normal state ν on $W^*(\hat{G}; \Omega^*)$ consider the map

$$T_\nu: C_0(G) \rightarrow C_r^*(\hat{G}; \Omega), \quad T_\nu(x) = (\nu \otimes \iota)((\hat{W}_\Omega \Omega)_{21}(1 \otimes x)(\hat{W}_\Omega \Omega)_{21}^*),$$

introduced in Section 3.1. It extends to a G -equivariant normal u.c.p. map $L^\infty(G) \rightarrow W^*(\hat{G}; \Omega)$; recall that we consider the action of G on $L^\infty(G)$ by right translations. Furthermore, since Ω is admissible by Proposition 2.3, as we already used in the proof of Proposition 3.4 the image of $C_0(G)$ under the map $x \mapsto (\hat{W}_\Omega \Omega)_{21}(1 \otimes x)(\hat{W}_\Omega \Omega)_{21}^*$ is a non-degenerate C^* -subalgebra of $M(C_r^*(\hat{G}; \Omega^*) \otimes C_r^*(\hat{G}; \Omega))$. This implies that T_ν maps $M(C_0(G)) = C_b(G)$ into $M(C_r^*(\hat{G}; \Omega))$ and the map $M(C_0(G)) \rightarrow M(C_r^*(\hat{G}; \Omega))$ is strictly continuous on bounded sets. It follows that for any state ω on $C_r^*(\hat{G}; \Omega)$ the positive linear functional ωT_ν on $C_0(G)$ is again a state.

Since T_ν is a G -equivariant normal u.c.p. map, if $f \in \text{dom } \varphi = L^\infty(G)_+ \cap L^1(G)$, then $T_\nu(f) \in \text{dom } \tilde{\varphi}$ and

$$\tilde{\varphi}(T_\nu(f)) = \varphi(f) = \int_G f(g) dg.$$

For the action of G on itself by right translations there is no distinction between integrability of an element $f \in C_0(G)_+$ in the von Neumann algebraic and the C^* -algebraic sense: both conditions are

equivalent to $f \in L^1(G)$. For $f \in C_0(G)_+ \cap L^1(G)$ and any state ω on $C_r^*(\hat{G}; \Omega)$ we then get

$$\int_G \omega(\beta_g(T_\nu(f)))dg = \int_G \omega T_\nu(f(\cdot g))dg = \varphi(f) = \tilde{\varphi}(T_\nu(f)),$$

so $T_\nu(f)$ is β -integrable. As we discussed in Section 3.1, the span of the spaces $T_\nu(C_0(G))$ over all normal states ν on $W^*(\hat{G}_\Omega; \Omega^*)$ is dense in $C_r^*(\hat{G}; \Omega)$. Hence $[\mathcal{P}_\beta^+] = C_r^*(\hat{G}; \Omega)$. \square

Returning to a general action γ of G on B , it is easy to see that if $b \in B_+$ is γ -integrable and $x \in M(B)^\gamma$, then x^*bx is again γ -integrable and $\psi_\gamma(x^*bx) = x^*\psi_\gamma(b)x$. This implies that the span of \mathcal{P}_γ^+ is an $M(B)^\gamma$ -bimodule, which in turn implies that the span of $\psi_\gamma(\mathcal{P}_\gamma^+)$ is a $*$ -ideal in $M(B)^\gamma$. The C*-algebra $[\psi_\gamma(\mathcal{P}_\gamma^+)] \subset M(B)^\gamma$ can be considered as a generalized fixed point algebra for the action γ on B . In general, however, it is too big to have good properties and it is not clear what the correct definition of a generalized fixed point algebra should be, see the discussion in [28].

Consider now a continuous action α of G on a C*-algebra A . Recall that by Proposition 3.4 the Ω -deformation A_Ω is a subalgebra of $M(C_r^*(\hat{G}; \Omega) \otimes A)^{\beta \otimes \alpha}$. We can now prove a slightly more precise result.

Proposition 4.2. *For any measurable unitary 2-cocycle Ω on \hat{G} and any C*-algebra A equipped with a continuous action α of G , the diagonal action $\beta \otimes \alpha$ of G on $C_r^*(\hat{G}; \Omega) \otimes A$ is integrable and the C*-algebra A_Ω is contained in $[\psi_{\beta \otimes \alpha}(\mathcal{P}_{\beta \otimes \alpha}^+)] \subset M(C_r^*(\hat{G}; \Omega) \otimes A)^{\beta \otimes \alpha}$.*

Proof. The first statement follows immediately from the integrability of β .

In order to prove the second statement denote by ρ the action of G on $C_0(G)$ by right translations. Then it is easy to check that for any $f \in C_0(G)_+ \cap L^1(G)$ and $a \in A_+$ the element $f \otimes a$ is $\rho \otimes \alpha$ -integrable and

$$\psi_{\rho \otimes \alpha}(f \otimes a) = \alpha(a_f),$$

where $a_f = \int_G f(g)\alpha_g(a)dg$; note that if we identify $M(C_0(G) \otimes A)$ with $C_b(G; M(A))$ then by definition $\alpha(a_f)(g) = \alpha_{g^{-1}}(a_f)$.

Now, as in the previous proof, consider the completely positive maps T_ν . Arguing as in that proof we conclude that the element $T_\nu(f) \otimes a$ is $\beta \otimes \alpha$ -integrable and

$$\psi_{\beta \otimes \alpha}(T_\nu(f) \otimes a) = (T_\nu \otimes \iota)\alpha(a_f).$$

But by (3.2) we have

$$(T_\nu \otimes \iota)\alpha(a_f) = (\nu \otimes \iota \otimes \iota)\eta_\Omega(\alpha(a_f)),$$

so the elements of the form $(T_\nu \otimes \iota)\alpha(a_f) \in \psi_{\beta \otimes \alpha}(\mathcal{P}_{\beta \otimes \alpha}^+)$ generate A_Ω . Since $[\psi_{\beta \otimes \alpha}(\mathcal{P}_{\beta \otimes \alpha}^+)]$ is a C*-algebra, it follows that A_Ω is contained in $[\psi_{\beta \otimes \alpha}(\mathcal{P}_{\beta \otimes \alpha}^+)]$. \square

In the case when G is a compact group all the analytical difficulties disappear and we get the following.

Proposition 4.3. *If G is compact, then $A_\Omega = (C_r^*(\hat{G}; \Omega) \otimes A)^{\beta \otimes \alpha}$.*

Proof. When G is compact, every element of $C_r^*(\hat{G}; \Omega) \otimes A$ is integrable and the map $\psi_{\beta \otimes \alpha}$ extends by linearity to a bounded map $C_r^*(\hat{G}; \Omega) \otimes A \rightarrow M(C_r^*(\hat{G}; \Omega) \otimes A)^{\beta \otimes \alpha}$ with image $(C_r^*(\hat{G}; \Omega) \otimes A)^{\beta \otimes \alpha}$. As follows from the previous proof, a dense subspace of elements of $C_r^*(\hat{G}; \Omega) \otimes A$ is mapped by $\psi_{\beta \otimes \alpha}$ onto a generating subspace of A_Ω . Hence $A_\Omega = (C_r^*(\hat{G}; \Omega) \otimes A)^{\beta \otimes \alpha}$. \square

For an overview of what is known about cocycles on duals of compact groups see [24].

4.3. Regularity of cocycles and proper actions. A stronger notion than integrability was introduced by Rieffel in [26]. Namely, an action γ of G on a C^* -algebra B is called proper, if there exists a dense γ -invariant $*$ -subalgebra $B_0 \subset B$ such that for all $b, c \in B$ the functions $g \mapsto b\gamma_g(c)$ and $g \mapsto \Delta_G(g)^{1/2}b\gamma_g(c)$ are norm-integrable* and there exists an element $x \in M(B_0)^\gamma \subset M(B)^\gamma$ such that $\int_G a\gamma_g(bc)dg = ax$ for all $a \in B_0$. By [28, Proposition 4.6] a proper action is integrable; in fact, if $b \in B_0$ then b^*b is γ -integrable.

The integrable functions $g \mapsto \Delta_G(g)^{1/2}b\gamma_g(c)$ define elements of the reduced crossed product $G \ltimes_\gamma B$. The closure I of the space spanned by such elements is a $*$ -ideal in $G \ltimes_\gamma B$. As shown in [26], this ideal is strongly Morita equivalent to the C^* -subalgebra $[\psi_\gamma(B_0^2)] \subset M(B)^\gamma$. The action γ is called saturated if B_0 can be chosen such that $I = G \ltimes_\gamma B$.

The relevance of these notions for us is explained by the following.

Proposition 4.4. *For a measurable unitary 2-cocycle Ω on \hat{G} , assume the action β of G on $C_r^*(\hat{G}; \Omega)$ is proper and saturated. Then Ω is regular.*

Proof. Since $M(C_r^*(\hat{G}; \Omega))^\beta = \mathbb{C}1$, by the above discussion the assumptions of the proposition imply that $C_r^*(\hat{G}) \rtimes_\beta G$ is strongly Morita equivalent to \mathbb{C} , that is, $C_r^*(\hat{G}; \Omega) \rtimes_\beta G$ is isomorphic to the algebra of compact operators on some Hilbert space. But this is one of the equivalent characterizations of regularity. \square

We expect the saturation property to hold more or less automatically. For example, it holds when G is compact, in which case, however, we do need the above proposition to show regularity.

We finish this section with a simple result on continuity of dual cocycles.

Proposition 4.5. *Assume Ω is a measurable unitary 2-cocycle on \hat{G} such that both Ω and Ω^* map $C_c(G \times G) \subset L^2(G \times G)$ into $L^1(G \times G) \cap L^2(G \times G)$. Then Ω is continuous, that is, $\Omega \in M(C_r^*(G) \otimes C_r^*(G))$.*

Proof. A function $f \in L^1(G \times G) \cap L^2(G \times G)$ defines both a vector in $L^2(G \times G)$ and an element $(\lambda \otimes \lambda)(f) \in C_r^*(G) \otimes C_r^*(G)$. We claim that if $f \in C_c(G \times G)$, then

$$\Omega(\lambda \otimes \lambda)(f) = (\lambda \otimes \lambda)(\Omega f).$$

Indeed, if $\xi \in C_c(G \times G)$, then, using that Ω commutes with the operator $\zeta \mapsto \zeta * \xi$ on $L^2(G \times G)$, we have

$$\Omega(\lambda \otimes \lambda)(f)\xi = \Omega(f * \xi) = (\Omega f) * \xi = (\lambda \otimes \lambda)(\Omega f)\xi.$$

Since $C_c(G \times G)$ is dense in $L^2(G \times G)$, this proves our claim.

It follows that $\Omega(C_r^*(G) \otimes C_r^*(G)) \subset C_r^*(G) \otimes C_r^*(G)$. Similarly, $\Omega^*(\lambda \otimes \lambda)(f) = (\lambda \otimes \lambda)(\Omega^* f)$ for all $f \in C_c(G \times G)$, which implies that $\Omega^*(C_r^*(G) \otimes C_r^*(G)) \subset C_r^*(G) \otimes C_r^*(G)$. \square

5. DUAL COCYCLES FOR A CLASS OF SOLVABLE LIE GROUPS

In this section we briefly consider dual cocycles recently constructed by Bieliavsky et al. [6, 5].

5.1. Deformation of negatively curved Kählerian Lie groups. As explained in [6], by results of Pyatetskii-Shapiro, negatively curved Kählerian Lie groups can be decomposed into iterated semidirect products of certain elementary groups, called elementary normal j -groups in [6]. To simplify matters we will consider only the latter groups. Thus throughout the whole Section 5 we assume that G is a simply connected real Lie group of dimension $2d + 2$ with a basis $H, \{X_j\}_{j=1}^{2d}, E$ of the Lie algebra \mathfrak{g} satisfying the relations

$$[H, E] = 2E, \quad [H, X_j] = X_j, \quad [E, X_j] = 0, \quad [X_i, X_j] = (\delta_{i+d, j} - \delta_{i, j+d})E.$$

*We define the modular function so that $\int_G f(gh)dg = \Delta_G(h)^{-1} \int_G f(g)dg$, which is opposite to the conventions in [26].

The map

$$\mathbb{R} \times \mathbb{R}^{2d} \times \mathbb{R} \ni (a, v, t) \mapsto \exp(aH) \exp \left(\sum_{j=1}^{2d} v_j X_j + tE \right) \in G$$

is a diffeomorphism. In the coordinates (a, v, t) the group law on G takes the form

$$(a, v, t)(a', v', t') = (a + a', e^{-a'}v + v', e^{-2a'}t + t' + \frac{1}{2}e^{-a'}\omega_0(v, v')),$$

where $\omega_0(v, v') = \sum_{i=1}^d (v_i v'_{i+d} - v_{i+d} v'_i)$ is the standard symplectic form on \mathbb{R}^{2d} . From this formula it is clear that the usual Lebesgue measure on \mathbb{R}^{2d+2} defines a left Haar measure on G . Then the modular function Δ_G is given by[†]

$$\Delta_G(a, v, t) = e^{-(2d+2)a}.$$

For every $\theta \in \mathbb{R}$, $\theta \neq 0$, Bieliavsky and Gayral [6, Section 4.1] construct a new product \star_θ on a space $\mathcal{E}_\theta(G)$ of smooth functions on G . The precise definition of $\mathcal{E}_\theta(G)$ is not important to us. What we need to know is that $\mathcal{E}_\theta(G)$ contains $C_c^\infty(G)$ and there exists a bijective linear map $T_\theta: S(\mathbb{R}^{2d+2}) \rightarrow \mathcal{E}_\theta(G)$ that is compatible with complex conjugation and that extends to a unitary operator $L^2(\mathbb{R}^{2d+2}) \rightarrow L^2(G)$. The new product is then defined by

$$f_1 \star_\theta f_2 = T_\theta(T_\theta^{-1}(f_1) \star_\theta^0 T_\theta^{-1}(f_2)),$$

where \star_θ^0 denotes the standard Moyal product on $S(\mathbb{R}^{2d+2})$ defined using the symplectic form

$$\omega_g((a, v, t), (a', v', t')) = 2(at' - ta') + \omega_0(v, v').$$

One of the reasons to introduce the map T_θ is that the product \star_θ becomes left G -invariant. Furthermore, it is possible to explicitly write down the distribution kernel of the product:

$$(f_1 \star_\theta f_2)(g) = \int_{G \times G} K_\theta(x, y) f_1(gx) f_2(gy) dx dy \quad \text{for } f_1, f_2 \in C_c^\infty(G),$$

where

$$K_\theta(x, y) = \frac{4}{(\pi\theta)^{2d+2}} A(x, y) \exp \left\{ \frac{2i}{\theta} S(x, y) \right\}$$

and, for $x = (a, v, t)$ and $x' = (a', v', t')$,

$$A(x, x') = (\cosh(a) \cosh(a') \cosh(a - a'))^d (\cosh(2a) \cosh(2a') \cosh(2a - 2a'))^{1/2},$$

$$S(x, x') = \sinh(2a)t' - \sinh(2a')t + \cosh(a) \cosh(a') \omega_0(v, v').$$

In view of our discussion of the relation between dual cocycles and deformations of the Fourier algebra in Section 4.1 it is then natural to try to define a cocycle Ω_θ on \hat{G} by

$$\Omega_\theta^* = \int_{G \times G} K_\theta(x, y) \lambda_x \otimes \lambda_y dx dy \quad \text{on } C_c^\infty(G \times G) \subset L^2(G \times G).$$

The question is whether this defines a unitary operator on $L^2(G \times G)$. By [5] this is indeed the case. In fact, the proof is essentially contained already in [6]. Namely, as established in the proof of [6, Proposition 8.45], we have

$$\int_{G \times G} K_\theta(x, y) K_{-\theta}(gx, hy) \Delta_G(x) \Delta_G(y) dx dy = \delta_e(g) \delta_e(h),$$

with the integral understood in the distribution sense. It is not difficult to check that this, together with $\bar{K}_\theta(x, y) = K_{-\theta}(x, y)$, implies that Ω_θ is an isometry. Similarly, we have

$$\int_{G \times G} K_{-\theta}(x, y) K_\theta(xg, yh) dx dy = \delta_e(g) \delta_e(h),$$

[†]Note again that our definition of the modular function is opposite to the one in [6].

which implies that Ω_θ^* is an isometry.

Therefore, by [5], we get a family of unitary 2-cocycles Ω_θ on \hat{G} . The corresponding quantum groups G_{Ω_θ} provide a non-formal deformation of the Poisson-Lie group G , with the Poisson structure defined by the non-degenerate 2-cocycle $\omega_{\mathfrak{g}}$ on \mathfrak{g} .

In the simplest case $d = 0$ the group G is the $ax + b$ group. A quantization of the same Poisson-Lie structure on it has been defined by Baaaj and Skandalis [30], see also [32, Section 5.3] (note also that there exists only one, up to isomorphism and rescaling, different Poisson-Lie structure; it has been quantized by Pusz, Woronowicz and Zakrzewski [25, 37]). It would be interesting to check whether for $d = 0$ the quantum groups G_{Ω_θ} are isomorphic to the one defined by Baaaj and Skandalis.

5.2. Involution on the twisted group algebra. Given the cocycle Ω_θ , we can now consider the new product \star_{Ω_θ} on the Fourier algebra $A(G)$ and the representation π_{Ω_θ} of $(A(G), \star_{\Omega_\theta})$ on $L^2(G)$ given by

$$\pi_{\Omega_\theta}(f_1)\check{f}_2 = (f_1 \star_{\Omega_\theta} f_2)^\sim \text{ for } f_1 \in A(G) \text{ and } f_2 \in A(G) \cap C_c(G).$$

At a first glance a bit surprisingly, the algebra $\pi_{\Omega_\theta}(A(G))$ fails to be a $*$ -algebra. Namely, consider the modular function Δ_G as the unbounded operator of multiplication by Δ_G on $L^2(G)$, so identify Δ_G with the modular operator defined by the Haar weight on $L^\infty(\hat{G}) = W^*(G)$. Consider also the dense subspace $A_\infty(G)$ of $A(G)$ spanned by the functions of the form $\xi * \zeta$ with $\xi, \zeta \in C_c^\infty(G)$. We then have the following.

Lemma 5.1. *For any $f \in A_\infty(G)$ we have*

$$\pi_{\Omega_\theta}(f)^* = \Delta_G^{-1} \pi_{\Omega_\theta}(\bar{f}) \Delta_G \text{ on } C_c(G) \subset L^2(G).$$

Proof. Since for $f \in A_\infty(G)$ we have

$$(f \otimes \iota)(\hat{W}(\lambda_x \otimes \lambda_y)) = \check{f}(x^{-1} \cdot) \lambda_y,$$

we get

$$\pi_{\Omega_\theta}(f) = \int_{G \times G} K_\theta(x, y) \check{f}(x^{-1} \cdot) \lambda_y \, dx \, dy \text{ on } C_c(G).$$

It follows that on $C_c(G)$ we have

$$\begin{aligned} \pi_{\Omega_\theta}(f)^* &= \int_{G \times G} \overline{K_\theta(x, y)} \lambda_{y^{-1}} \check{f}(x^{-1} \cdot) \, dx \, dy = \int_{G \times G} \overline{K_\theta(x, y)} \check{f}(x^{-1} y \cdot) \lambda_{y^{-1}} \, dx \, dy \\ &= \int_{G \times G} \overline{K_\theta(yx, y)} \check{f}(x^{-1} \cdot) \lambda_{y^{-1}} \, dx \, dy = \int_{G \times G} \overline{K_\theta(y^{-1}x, y^{-1})} \Delta_G(y)^{-1} \check{f}(x^{-1} \cdot) \lambda_y \, dx \, dy \\ &= \Delta_G^{-1} \int_{G \times G} \overline{K_\theta(y^{-1}x, y^{-1})} \check{f}(x^{-1} \cdot) \lambda_y \, dx \, dy \, \Delta_G. \end{aligned}$$

Therefore it suffices to check that $\overline{K_\theta(y^{-1}x, y^{-1})} = K_\theta(x, y)$, or equivalently,

$$A(y^{-1}x, y^{-1}) = A(x, y) \text{ and } S(y^{-1}x, y^{-1}) = -S(x, y).$$

Both identities are checked by a straightforward computation. \square

We will now give a different proof of this lemma using known properties of the Moyal product. As we discussed in Section 4.1, the representation π_{Ω_θ} can be thought of as the left regular representation of $(A(G), \star_{\Omega_\theta})$ with respect to the scalar product defined by the right Haar measure. We can also try to use the left Haar measure. In general we see no reason to expect the corresponding representation to be well-defined. But in the present case, where \star_θ was constructed using the Moyal product, we do have a representation π_θ of $(\mathcal{E}_\theta(G), \star_\theta)$ on $L^2(G)$ defined by

$$\pi_\theta(f_1)f_2 = f_1 \star_\theta f_2 \text{ for } f_1, f_2 \in \mathcal{E}_\theta(G).$$

Furthermore, for this representation we have $\pi_\theta(f)^* = \pi_\theta(\bar{f})$. Since by construction the products \star_θ and \star_{Ω_θ} coincide on $A_\infty(G) \subset A(G) \cap \mathcal{E}_\theta(G)$ (but $A_\infty(G)$ is not closed under these products), for any $f \in A_\infty(G)$ we have

$$\pi_{\Omega_\theta}(f) = \nabla \pi_\theta(f) \nabla \quad \text{on } A_\infty(G) \subset L^2(G),$$

where ∇ is the unbounded involutive operator defined by $\nabla f = \check{f}$. Since $\nabla^* = \Delta_G^{-1} \nabla = \nabla \Delta_G$, this is consistent with Lemma 5.1.

In view of the identity $\pi_{\Omega_\theta}(f) = \nabla \pi_\theta(f) \nabla$ it may seem surprising that both representations π_{Ω_θ} and π_θ are well-defined. The representation π_{Ω_θ} is well-defined by our general theory. The reason why π_θ is well-defined is that ultimately the product \star_θ was constructed using a dual cocycle on \mathbb{R}^{2d+2} and this group is unimodular.

Turning to C*-algebras, the obvious conclusion is that the identity map on $A_\infty(G)$ does not extend to a *-isomorphism of $C_r^*(\hat{G}; \Omega_\theta) = \overline{\pi_{\Omega_\theta}(A(G))}$ and $\overline{\pi_\theta(\mathcal{E}_\theta(G))}$. This, however, does not exclude the possibility that these C*-algebras are isomorphic in a canonical G -equivariant way. In fact, the above considerations suggest that the conjugation by the involutive unitary $\Delta_G^{-1/2} \nabla = J\hat{J}$ gives such an isomorphism. This will be analyzed in a subsequent publication.

The C*-algebra $\overline{\pi_\theta(\mathcal{E}_\theta(G))}$ is the θ -deformation of $C_0(G)$ as defined by Bieliavsky and Gayral [6]. To be more precise, instead of the representation π_θ they use the Weyl quantization map. But it is well-known that this gives a quasi-equivalent representation. In particular, as an abstract C*-algebra, $\overline{\pi_\theta(\mathcal{E}_\theta(G))}$ is isomorphic to the algebra of compact operators on an infinite dimensional separable Hilbert space.

5.3. Two-parameter deformation. The papers [6] and [5] contain a more general class of deformations, with a second parameter of deformation being a function on \mathbb{R} . These deformations are obtained by inserting an additional factor into the definition of the map T_θ . We do not need the precise definition of this procedure, see [6, Section 4.1] for details, and will only write down the final answer.

Given a smooth function τ on \mathbb{R} satisfying certain growth conditions, we have a G -invariant product $\star_{\theta,\tau}$ on a function space $\mathcal{E}_{\theta,\tau}(G)$ defined by the kernel

$$K_{\theta,\tau}(x, x') = K_\theta(x, x') \exp \left\{ \tau \left(\frac{2}{\theta} \sinh(2a) \right) + \tau \left(\frac{2}{\theta} \sinh(-2a') \right) - \tau \left(\frac{2}{\theta} \sinh(2a - 2a') \right) \right\}.$$

If τ is purely imaginary, this kernel defines a unitary 2-cocycle $\Omega_{\theta,\tau}$ on \hat{G} such that

$$\Omega_{\theta,\tau}^* = \int_{G \times G} K_{\theta,\tau}(x, y) \lambda_x \otimes \lambda_y dx dy \quad \text{on } C_c^\infty(G \times G) \subset L^2(G \times G).$$

In order to understand this cocycle, consider the von Neumann algebra $W^*(\mathbb{R})$ of \mathbb{R} . The conjugation by the inverse of the Fourier transform, defined by

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\xi t} dt,$$

gives an isomorphism $L^\infty(\mathbb{R}) \cong W^*(\mathbb{R})$, so the unitary $e^{-\tau} \in L^\infty(\mathbb{R})$ defines a unitary $u_\tau = \mathcal{F}^{-1} e^{-\tau} \mathcal{F} \in W^*(\mathbb{R})$. Using the embedding $\mathbb{R} \hookrightarrow G$, $t \mapsto (0, 0, t)$, we get an embedding $W^*(\mathbb{R}) \hookrightarrow W^*(G)$, so we can consider the unitary u_τ as an element of $W^*(G)$.

Proposition 5.2. *We have $\Omega_{\theta,\tau} = (u_\tau \otimes u_\tau) \Omega_\theta \hat{\Delta}(u_\tau)^*$, so the cocycles $\Omega_{\theta,\tau}$ and Ω_θ are cohomologous.*

Proof. It is convenient to prove a slightly different statement. Fix functions $f_1, f_2 \in S(\mathbb{R})$. Consider the corresponding elements $b_i = \mathcal{F}^{-1} f_i \mathcal{F}$ of $W^*(\mathbb{R}) \subset W^*(G)$. Explicitly,

$$b_i = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{f}_i(-t) \lambda_{(0,0,t)} dt.$$

Consider also the function

$$\tilde{K}(x, x') = K_\theta(x, x') f_1 \left(\frac{2}{\theta} \sinh(2a) \right) f_1 \left(\frac{2}{\theta} \sinh(-2a') \right) f_2 \left(\frac{2}{\theta} \sinh(2a - 2a') \right)$$

and the operator $\tilde{\Omega}^* = \int_{G \times G} \tilde{K}(x, y) \lambda_x \otimes \lambda_y dx dy$, which is at least defined on $C_c^\infty(G \times G)$. We claim that this is a bounded operator on $L^2(G \times G)$ and

$$\tilde{\Omega}^* = \hat{\Delta}(b_2) \Omega_\theta^*(b_1 \otimes b_1). \quad (5.1)$$

Denote $4(\pi\theta)^{-2d-2}A(x, x')$ by $\tilde{A}(a, a')$ (recall that $A(x, x')$ depends only on the coordinates a, a'). Then on $C_c^\infty(G \times G)$ we have:

$$\begin{aligned} \tilde{\Omega}^* &= \int \tilde{A}(a, a') \exp \left\{ \frac{2i}{\theta} S(x, x') \right\} f_1 \left(\frac{2}{\theta} \sinh(2a) \right) f_1 \left(\frac{2}{\theta} \sinh(-2a') \right) f_2 \left(\frac{2}{\theta} \sinh(2a - 2a') \right) \\ &\quad \times \lambda_x \otimes \lambda_{x'} dx dx' \\ &= \frac{1}{(2\pi)^{3/2}} \int \hat{f}_1(t_1) \hat{f}_1(t'_1) \hat{f}_2(t_2) \tilde{A}(a, a') \exp \left\{ \frac{2i}{\theta} \cosh(a) \cosh(a') \omega_0(v, v') \right\} \\ &\quad \times \exp \left\{ \frac{2i}{\theta} (t_1 \sinh(2a) - t'_1 \sinh(2a') + t_2 (\sinh(2a) \cosh(2a') - \cosh(2a) \sinh(2a'))) \right\} \\ &\quad \times \exp \left\{ \frac{2i}{\theta} (\sinh(2a)t' - \sinh(2a')t) \right\} \lambda_x \otimes \lambda_{x'} dt_1 dt'_1 dt_2 dx dx' \\ &= \frac{1}{(2\pi)^{3/2}} \int \hat{f}_1(t_1) \hat{f}_1(t'_1) \hat{f}_2(t_2) \tilde{A}(a, a') \exp \left\{ \frac{2i}{\theta} \cosh(a) \cosh(a') \omega_0(v, v') \right\} \\ &\quad \times \exp \left\{ \frac{2i}{\theta} (\sinh(2a)t' - \sinh(2a')t) \right\} \\ &\quad \times \lambda_{(a, v, t - t'_1 - t_2 \cosh(2a))} \otimes \lambda_{(a', v', t' - t_1 - t_2 \cosh(2a'))} dt_1 dt'_1 dt_2 dx dx'. \end{aligned}$$

The group multiplication formula gives

$$(a, v, t - t'_1 - t_2 \cosh(2a)) = (0, 0, -t_2)(a, v, t - t_2 \sinh(2a))(0, 0, -t'_1).$$

Hence

$$\begin{aligned} \tilde{\Omega}^* &= \frac{1}{(2\pi)^{3/2}} \int \hat{f}_1(t_1) \hat{f}_1(t'_1) \hat{f}_2(t_2) \tilde{A}(a, a') \exp \left\{ \frac{2i}{\theta} \cosh(a) \cosh(a') \omega_0(v, v') \right\} \\ &\quad \times \exp \left\{ \frac{2i}{\theta} (\sinh(2a)(t' + t_2 \sinh(2a')) - \sinh(2a')(t + t_2 \sinh(2a))) \right\} \\ &\quad \times (\lambda_{(0, 0, -t_2)} \otimes \lambda_{(0, 0, -t_2)})(\lambda_x \otimes \lambda_{x'})(\lambda_{(0, 0, -t'_1)} \otimes \lambda_{(0, 0, -t_1)}) dt_1 dt'_1 dt_2 dx dx' \\ &= \hat{\Delta}(b_2) \Omega_\theta^*(b_1 \otimes b_1). \end{aligned}$$

Now, choosing a bounded sequence of functions $g_n \in S(\mathbb{R})$ converging to e^τ pointwise and passing to the limit in identity (5.1) applied to the pairs $(f_1, f_2) = (g_n, \bar{g}_n)$, we get the result. \square

Therefore, from our perspective, there is no reason to introduce the second deformation parameter τ , since by Proposition 3.12 this leads to isomorphic deformations. Note also that on the level of the function spaces $\mathcal{E}_{\theta, \tau}(G)$ the corresponding G -equivariant isomorphism $(\mathcal{E}_\theta(G), \star_\theta) \cong (\mathcal{E}_{\theta, \tau}(G), \star_{\theta, \tau})$ is given by the operator $\mathcal{F}^{-1} e^{-\tau} \mathcal{F}$.

6. OPEN PROBLEMS

The results that we have obtained so far lead to a number of questions that have to be resolved in order to bring the theory to a completely satisfactory level. In this section we list some of the most natural ones.

6.1. Admissibility. As we already mentioned in Section 2.1, we find it plausible that any cocycle is admissible. While the question whether this is true is interesting, it is however not the most pressing one, since at least for regular quantum groups any cocycle is indeed admissible.

6.2. Regularity of cocycles. The problem is to find simple, verifiable conditions for regularity. In view of our considerations in Sections 4.2 and 4.3 for group duals, it seems unlikely that such conditions exist. At the same time we do not have a single example of a non-regular cocycle on a regular quantum group.

6.3. Regularity of deformed quantum groups. As has been shown by De Commer [10], regularity of a quantum group is not preserved under cocycle deformation: the non-regular quantum group $\tilde{E}_q(2)$ is obtained by deformation by a cocycle on $SU_q(2)$. In this respect we want to formulate the following question: if G is a regular quantum group and Ω is a cocycle on \hat{G} , is regularity of Ω equivalent to regularity of G_Ω ? If not, do we have an implication in at least one direction?

6.4. Generalized fixed point algebras. By Proposition 4.2, in the case of cocycles on group duals, for the Ω -deformation A_Ω of a C*-algebra A we have $A_\Omega \subset [\psi_{\beta \otimes \alpha}(\mathcal{P}_{\beta \otimes \alpha}^+)]$. It is not difficult to see that the inclusion can be strict already for $G = \mathbb{Z}$ and $\Omega = 1$. What is the proper characterization of elements of A_Ω in terms of the action $\beta \otimes \alpha$ of G on $C_r^*(\hat{G}; \Omega) \otimes A$?

Another question is what an analogue of this setting for general quantum groups is. When G is a genuine group, what is of course special, is that any action of G can be viewed as a left or a right action and the tensor product action is always well-defined. When G is a group dual, then again we can always pass from a left to a right action. But in order to define the diagonal action we have to replace the usual tensor product by the braided tensor product \boxtimes . As was shown by Yamashita [38], when in addition G is compact, so $C_0(G) = C_r^*(\Gamma)$ for a discrete group Γ , then A_Ω is isomorphic to the fixed point algebra $(C_r^*(\Gamma; \Omega) \boxtimes A)^G$. But for general quantum groups, when $(C_0(G), \Delta)$ is neither commutative nor cocommutative, it is not clear to us what the correct analogue of the description of A_Ω as (a subalgebra of) a fixed point algebra is.

6.5. Generalization of Rieffel's deformation. In the setting of Section 5, Bieliavsky and Gayral defined a θ -deformation A_θ of any C*-algebra A equipped with an action of G . The question is how the algebras A_θ are related to our algebras A_{Ω_θ} . As we have seen, this question is not quite trivial already for $A = C_0(G)$. Assuming it can be rigorously settled in this case, for general A both algebras A_θ and A_{Ω_θ} can be embedded into $M(C_r^*(\hat{G}; \Omega_\theta) \otimes A)^{\beta \otimes \alpha}$, and then the question is whether they coincide. The analogous question for Rieffel's deformation has an affirmative answer [3] (although this is not how the result was formulated in [3]). There are several reasons why it will be difficult to give a similar proof in the present case. One of them is that in the case of Rieffel's deformation the group \mathbb{R}^{2d} carrying a dual cocycle was abelian, so the deformation still carried an action of the same group. This is no longer the case for the groups considered by Bieliavsky and Gayral, where we can only hope that A_θ carries an action of G_{Ω_θ} .

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